

ALGORITHMIC ASPECTS OF REGULAR GRAPH COVERS*

JIRÍ FIALA[†], PAVEL KLAVÍK[‡], AND JAN KRATOCHVÍL[†] AND ROMAN NEDELA^{§,¶}

Abstract. A graph G *covers* a graph H if there exists a locally bijective homomorphism from G to H . We deal with *regular covers* where this homomorphism is prescribed by the action of a semiregular subgroup of $\text{Aut}(G)$. We study *computational aspects* of regular covers that have not been addressed before. The decision problem **REGULARCOVER** asks for given graphs G and H whether G regularly covers H . When $|H| = 1$, this problem becomes Cayley graph recognition for which the complexity is still unresolved. Another special case arises for $|G| = |H|$ when it becomes the graph isomorphism problem.

Our main result is an involved FPT algorithm solving **REGULARCOVER** for planar inputs G in time $\mathcal{O}^*(2^{e(H)/2})$ where $e(H)$ denotes the number of edges of H . The algorithm is based on dynamic programming and employs theoretical results proved in a related structural paper. Further, when G is 3-connected, H is 2-connected or the ratio $|G|/|H|$ is an odd integer, we can solve the problem **REGULARCOVER** in polynomial time. In comparison, Bílka et al. (2011) proved that testing general graph covers is NP-complete for planar inputs G when H is a small fixed graph such as K_4 or K_5 .

Key words. regular graph covers, planar graphs, FPT algorithm, computational complexity, graph isomorphism problem, Cayley graph recognition

1. Introduction. The notion of *covering* originates in topology as a notion of local similarity of two topological spaces. For instance, consider the unit circle and the real line. Globally, these two spaces are not the same, they have different properties, different fundamental groups, etc. But when we restrict ourselves to a small part of the circle, it looks the same as a small part of the real line; more precisely the two spaces are locally homeomorphic, and thus they share the local properties. The notion of covering formalizes this property of two spaces being *locally the same*.

Suppose that we have two topological spaces: a big one G and a small one H . We say that G *covers* H if there exists an epimorphism called a *covering projection* $p : G \rightarrow H$ which locally preserves the structure of G . For instance, the mapping $p(x) = (\cos x, \sin x)$ from the real line to the unit circle is a covering projection. The existence of a covering projection ensures that G looks locally the same as H ; see Fig. 1.1a.

In this paper, we study coverings of graphs in a more restricting version called *regular covering*, for which the covering projection is described by an action of a group; see Section 2 for the formal definition. If G regularly covers H , then we say that H is a (*regular*) *quotient* of G .

Negami's Theorem [52], stating that all regular quotients of planar graphs can be embedded into the projective plane, is one of the oldest results in topological graph theory. Therefore, we have decided to initiate the study of computational complexity of regular graph covers with planar graphs.

1.1. Applications of Graph Coverings. Suppose that G covers H and we have some information about one of the objects. How much knowledge does translate to the other object? It turns out that quite a lot, and this makes covering a powerful technique with many diverse applications. The big advantage of regular coverings is that they can be efficiently described and many properties easily translate between the objects. We sketch some applications now.

Powerful Constructions. The reverse of covering called *lifting* can be applied to small objects in order to construct large objects of desired properties. For instance, the well-known Cayley graphs are large objects which can be described easily by a few elements of a group. Let G be a Cayley graph generated by elements g_1, \dots, g_e of a group Γ . The vertices of G correspond to the elements of Γ and the edges are described by

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[†]Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Malostranské náměstí 25, 118 00 Prague, Czech Republic.
E-mails: {fiala,honza}@kam.mff.cuni.cz.

[‡]Computer Science Institute, Faculty of Mathematics and Physics, Charles University, Malostranské náměstí 25, 118 00 Prague, Czech Republic. E-mail: klavik@iuuk.mff.cuni.cz.

[§]Institute of Mathematics and Computer Science SAS, Ľuberská 1, 974 11 Banská Bystrica, Slovak republic. Email: nedela@savbb.sk.

[¶]European Centre of Excellence NTIS, University of West Bohemia, Pilsen, Czech Republic.

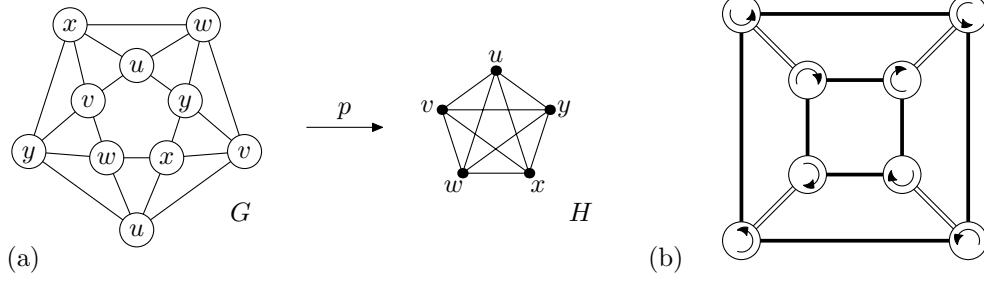


FIG. 1.1. (a) A covering projection p from a graph G to a graph H . (b) The Cayley graph of the dihedral group \mathbb{D}_4 generated by the 90° rotations (in black) and the reflection around the x -axis (in white).

actions of g_1, \dots, g_e on Γ by left multiplication; each g_i defines a permutation on Γ and we put edges along the cycles of this permutation. See Fig. 1.1b for an example. Cayley graphs were originally invented to study the structure of groups [16].

In the language of coverings, every Cayley graph G can be described as a lift of a one vertex graph H with e loops and half-edges attached labeled g_1, \dots, g_e . Regular covers can be viewed as a generalization of Cayley graphs where the small graph H can contain more than one vertex. For example, the famous Petersen graph can be constructed as a lift of a two-vertex graph H in Fig. 1.2a. These two vertices are necessary as it is known that Petersen graph is not a Cayley graph. Figure 1.2b shows a simple construction [50, 58] of the Hoffman-Singleton graph [35] which is a 7-regular graph with 50 vertices.

The Petersen and the Hoffman-Singleton graphs are extremal graphs for the *degree-diameter* problem: given integers d and k , find a maximal graph G with diameter d and degree k . In general, the size of G is not known. Many currently best constructions were obtained using the covering techniques [51].

Further applications employ the fact that nowhere-zero flows, vertex and edge colorings, eigenvalues and other graph invariants lift along a covering projection. Two main applications of constructions of lifts are the solution of the Heawood map coloring problem [53, 31] and constructions of arbitrarily large highly symmetrical graphs [10].

Models of Local Computation. These and similar constructions have many practical applications in designing highly efficient computer networks [20, 2, 9, 13, 14, 15, 32, 60], since these networks can be efficiently described/constructed and have many strong properties. In particular, networks based on covers of simple graphs allow fast parallelization of computation as described e.g. in [12, 3, 4].

Simplifying Objects. Regular coverings can be also applied in the opposite way, to project big objects onto smaller ones while preserving some properties. One way is to represent a class of objects satisfying some properties as quotients of the universal object of this property. For instance, this was used in the study of arc-transitive cubic graphs [30], and the key point is that universal objects are much easier to work with. This idea is commonly used in fields such as the theory of Riemann surfaces [22] and theoretical physics [40].

1.2. Regular Covering Testing. Despite all described applications, the computational complexity of regular covering was not yet studied. In this paper, we initiate the study of the following computational problem.

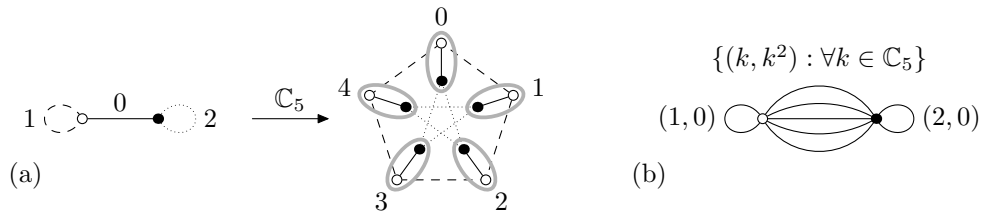


FIG. 1.2. (a) A construction of the Petersen graph by lifting with the group \mathbb{C}_5 . (b) By lifting the described graph with the group \mathbb{C}_5^2 , we get the Hoffman-Singleton graph. The five parallel edges are labeled $(0, 0)$, $(1, 1)$, $(2, 4)$, $(3, 4)$ and $(4, 1)$.

Problem:	REGULARCOVER
Input:	Connected graphs G and H .
Output:	Does G regularly cover H ?

For a fixed graph H , the computational complexity of REGULARCOVER was first asked as an open problem by Abello et al. [1]: “Are there graphs H for which the problem of determining if an input graph G is a regular cover of H is NP-hard?” Currently, no NP-hardness reduction is known for REGULARCOVER, even when H is a part of the input. Our main result shows that if G is planar, no such graph H exists. We use the complexity notation $f = \mathcal{O}^*(g)$ which omits polynomial factors. We establish the following FPT algorithm:

THEOREM 1.1. *For planar graphs G , the REGULARCOVER problem can be solved in time $\mathcal{O}^*(2^{e(H)/2})$, where $e(H)$ is the number of edges of H .*

1.3. Related Computational Problems. We discuss other computational problems related to REGULARCOVER. The notion of regular covers builds a bridge between two seemingly different problems: Cayley graph recognition and the graph isomorphism problem.

Covering Testing. The complexity of general covering was widely studied before, pioneered by Bodlaender [12] in the context of networks of processors in parallel computing. Abello et al. [1] introduced the H -COVER problem which asks for an input graph G whether it covers a fixed graph H . Unless H is very simple, the problem turned out to be mostly NP-complete, the general complexity is still unresolved but the papers [45, 24] show that it is NP-complete for every r -regular graph H where $r \geq 3$. For a survey of the complexity results, see [27].

We try to understand how much the additional algebraic structure of regular covering changes the computational complexity. For planar inputs G , the change is significant: the problem H -COVER remains NP-complete for several small fixed graphs H (such as K_4 , K_5) [11], while REGULARCOVER can be solved in polynomial time for every fixed graph H by Theorem 1.1.

Cayley Graphs Testing. If the graph H consists of a single vertex with attached loops and half-edges, it corresponds to Cayley graph recognition whose computational complexity is widely open. No hardness results are known and a polynomial-time algorithm is known only for recognition of circulant graphs [21]. In contrast, if H consists of a vertex with three half-edges attached, then G covers H if and only if G is a cubic 3-edge-colorable graph, so H -COVER is NP-complete [36].

The reader may notice that Theorem 1.1 gives a polynomial time algorithm to recognize planar Cayley graphs. The input is a k -regular planar graph G , for $k \leq 5$. We test REGULARCOVER for all graphs H which a single vertex of degree k . Unfortunately, finite planar Cayley graphs G are very limited: either G is a cycle, or G is 3-connected. Therefore, $\text{Aut}(G)$ is a *spherical group* which is very simple. Therefore, G is either finite (with $v(G) \leq 120$), representing one of the sporadic groups (for instance, a truncated dodecahedron is a Cayley graph of A_5), or very simple (a cycle, a prism, an antiprism, e.g.).

Graph Isomorphism Problem. The other extreme is when both graphs G and H have the same size, for which REGULARCOVER is the famous graph isomorphism problem (GRAPHISO). Graph isomorphism belongs to NP, it is unlikely NP-complete, however no polynomial-time algorithm is known and Babai [8] recently proved that it can be solved in quasipolynomial time. Also, polynomial-time algorithms for GRAPHISO are known for many graph classes and parameters; see [41] for an overview. Since REGULARCOVER generalizes GRAPHISO, we cannot hope to solve it in polynomial time (unless solving GRAPHISO as well). It is natural to ask which results and techniques for GRAPHISO translate to REGULARCOVER. Our results show that some technique for planar graphs translate, but the REGULARCOVER problem is significantly more involved.

Theoretical motivation for studying the graph isomorphism problem is very similar to REGULARCOVER. For practical instances, one can solve GRAPHISO very efficiently using various heuristics. But polynomial-time algorithm working for all graphs is not known and it is very desirable to understand the complexity of GRAPHISO. It is known that testing graph isomorphism is equivalent to testing isomorphism of general mathematical structures [33]. The notion of isomorphism is widely used in mathematics when one wants to show that two seemingly different structures are the same. One proceeds by guessing a mapping and proving that this mapping is an isomorphism. The natural complexity question is whether there is a better algorithmic way to derive an isomorphism. Similarly, regular covering is a well-known mathematical notion which is algorithmically interesting and not understood.

Computing Automorphism Groups. A regular covering is described by a semiregular subgroup of the automorphism group $\text{Aut}(G)$. We denote the computational problem of finding generators of $\text{Aut}(G)$ by **AUTGROUP**. Since a good understanding of $\text{Aut}(G)$ is needed to solve **REGULARCOVER**, it is closely related to **AUTGROUP**.

It is known that **GRAPHISO** can be reduced to **AUTGROUP** (which forms the foundation of group theory techniques used to attack the graph isomorphism problem, e.g., [48, 8]), moreover **AUTGROUP** can be solved by $\mathcal{O}(n^3)$ instances of **GRAPHISO** [49]. Surprisingly not much is known about automorphism groups of restricted classes of graphs. Jordan [39] gave an inductive characterization of automorphism groups of trees as the class of groups closed under direct product and wreath product with symmetric groups. Babai [5, 7] described automorphism groups of planar graphs. Recently, Jordan-like characterizations of automorphism groups of interval, circle and permutation graphs are given in [43, 44]. The **AUTGROUP** problem can be solved in linear time for trees and interval graphs [18], in linear time for permutation graphs [44], and in polynomial time for circle graphs [44].

Our description of semiregular actions on planar graphs in [26] was generalized in [42] to describe a Jordan-like characterization of automorphism groups of planar graphs, which is much more detailed than Babai's description in [5]. It also implies a quadratic-time algorithm for **AUTGROUP** of planar graphs (which likely can be improved to linear time), faster than the best previous trivial $\mathcal{O}(n^4)$ algorithm by combining [38, 49].

List Restricted Isomorphism Problem. Let G and H be graphs and the vertices of G be *equipped by lists*: for each $u \in V(G)$, we have $\mathcal{L}(u) \subseteq V(H)$. An isomorphism $\pi : G \rightarrow H$ is called *list-compatible* if for every $u \in V(G)$, we have $\pi(u) \in \mathcal{L}(u)$. The existence of a list-compatible isomorphism is denoted by $G \xrightarrow{\mathcal{L}} H$.

Problem: LISTISO
Input: Graph G and H , and for each $u \in V(G)$ a list $\mathcal{L}(u) \subseteq V(H)$.
Output: Does $G \xrightarrow{\mathcal{L}} H$?

This problem was first introduced by Lubiw [47] and proved to be **NP**-complete, even in the following restricted setting.

THEOREM 1.2 (Lubiw [47]). *Testing existence of a fixed-point free involutory automorphism is NP-complete.*

But only the above result of [47] is cited while the LISTISO problem was forgotten. We have rediscovered LISTISO since it was solved in a subroutine in our algorithm of Theorem 1.1 for 3-connected planar and projectively planar graphs, for which it can be solved in polynomial time using [46]; see [25]. Our paper gives a nice motivation for LISTISO, leading Klavík et al. [41] to study it for many restricted graph classes and parameters. In particular, LISTISO can be solved in polynomial time for graphs of bounded genus and bounded treewidth [41].

We also consider special instances called **COLORISO** in which both graphs G and H are colored and we ask for existence of a color-preserving isomorphism, denoted $G \xrightarrow{c} H$. Unlike LISTISO, the **COLORISO** problem is a well known problem which is polynomial-time equivalent to **GRAPHISO**.

Homomorphisms and CSP. Since regular covering is a locally bijective homomorphism, we give an overview of complexity results concerning homomorphisms. Hell and Nešetřil [34] studied the problem H -HOM which asks whether there exists a homomorphism between an input graph G and a fixed graph H . Their celebrated dichotomy result for simple graphs states that the problem H -HOM is polynomially solvable if H is bipartite, and it is **NP**-complete otherwise. Homomorphisms can be described in the language of constraint satisfaction (CSP), and the famous dichotomy conjecture [23] claims that every CSP is either polynomially solvable, or **NP**-complete.

1.4. Other Covering Problems. We introduce and discuss several other problems related to (regular) graph covering.

Lifting and Quotients. In the **REGULARCOVER** problem, the input gives two graphs G and H . For the following problems, the input specifies only one graph and we ask for existence of the other graph:

Problem:	REGULARLIFTING
Input:	A connected graph H and an integer k .
Output:	Does there exists a graph G regularly covering H such that $ G = k H $?

Problem:	REGULARQUOTIENT
Input:	A connected graph G and an integer k .
Output:	Does there exists a graph H regularly covered by G such that $ H = \frac{ G }{k}$?

Concerning REGULARLIFTING, the answer is always positive. The theory of covering describes a technique called voltage assignment which can be applied to generate all k -folds G . We do not deal with lifting in this paper, but there are nevertheless many interesting computational questions with applications. For instance, is it possible to generate efficiently all (regular) lifts up to isomorphism? (This is non-trivial since different voltage assignments might lead to isomorphic graphs.) Or, does there exists a lift with some additional properties?

Concerning REGULARQUOTIENT, by Theorem 1.2, this problem is NP-complete even for the fixed $k = 2$. (We ask for existence of a half-quotient H of G which is equivalent to existence of a fixed-point free involution in $\text{Aut}(G)$.) This hardness reduction can be easily generalized for every fixed even k , but the complexity remains open for odd values of k .

The reduction of Theorem 1.2 is from 3-satisfiability, each variable is represented by a variable gadget which is an even cycle attached to the rest of the graph. Each cycle has two possible regular quotients, either the cycle of half length (obtained by the 180° rotation), or the path of half length with attached half-edges (obtained by a reflection through opposite edges), corresponding to true and false values, respectively. These variable gadgets are attached to clause gadgets, and a quotient of a clause gadget can be constructed if and only if at least one literal of the clause is satisfied. This reduction does not imply NP-completeness for the REGULARCOVER problem since the input also gives a graph H , so one can decode the assignment of the variables from it.

k -Fold Covering. To simplify the REGULARCOVER problem, instead of fixing H , we can fix the ratio $k = |G|/|H|$. (When G covers H , then k is an integer.) We get the following two problems for general and regular graph covers, respectively:

Problem:	k -FOLD(REGULAR)COVER
Input:	Connected graphs G and H such that $ G = k H $.
Output:	Does G (regularly) cover H ?

For $k = 1$, both problems are equivalent to GRAPHISO. Bodlaender [12] proved that the k -FOLDCOVER problem is GI-hard for every fixed k (meaning that GRAPHISO can be reduced to it). The same reduction also works for k -FOLDREGULARCOVER, see Lemma 2.3. Chaplick et al. [17] proved NP-completeness of 3-FOLDCOVER and their reduction can be easily modified for all $k > 3$. The complexities of 2-FOLDCOVER and k -FOLDREGULARCOVER for all $k \geq 2$ are open and very interesting. We note that for $k = 2$, every covering is a regular covering, so the problems 2-FOLDREGULARCOVER and 2-FOLDCOVER are identical, and NP-hardness of 2-FOLDCOVER would imply NP-hardness for REGULARCOVER as well. On the other hand, if k -FOLDREGULARCOVER is not NP-complete for any value k , the k -FOLDREGULARCOVER problems would be natural generalizations of GRAPHISO.

1.5. Three Properties. Let \mathcal{C} be a class of connected multigraphs. By \mathcal{C}/Γ we denote the class of all regular quotients of graphs of \mathcal{C} (note that $\mathcal{C} \subseteq \mathcal{C}/\Gamma$). For instance, when \mathcal{C} is the class of planar graphs, then the class \mathcal{C}/Γ is, by Negami Theorem [52], the class of projective planar graphs. We define the following three properties of \mathcal{C} , for formal definitions see Section 2:

- (P1) The classes \mathcal{C} and \mathcal{C}/Γ are closed under taking subgraphs and under replacing connected components attached to 2-cuts by edges.
- (P2) For a 3-connected graph $G \in \mathcal{C}$, all semiregular subgroups Γ of $\text{Aut}(G)$ can be computed in polynomial time. Here by semiregularity, we mean that the action of Γ has no non-trivial stabilizers of the vertices.
- (P3) Let G and H be 3-connected graphs of \mathcal{C}/Γ , possibly with colored and directed edges, and the vertices

of G be equipped with lists. We can decide LISTISO of G and H in polynomial time. (Where the list-compatible isomorphism respects orientations and colors of edges.)

As we prove in Lemma 6.2, these three properties are tailored for the class of planar graphs. (The proof of the property (P3) is non-trivial, following from [41].) The main reason to state (P1) to (P3) is explicitly to make clear which properties of planar graphs are necessary for our algorithm.

Since LISTISO is NP-complete in general, we also use the restricted version with only COLORISO to highlight places where LISTISO can be avoided:

(P3*) Let G and H be 3-connected graphs of \mathcal{C}/Γ , possibly with colored and directed edges, and the vertices of G and H are colored. We can decide COLORISO of G and H in polynomial time.

1.6. The Meta-algorithm. This paper studies complexity of regular covering testing, based on our structural results described in [26]. We establish the following algorithmic result:

THEOREM 1.3. *Let \mathcal{C} be a class of graphs satisfying (P1) to (P3). There exists an FPT algorithm for REGULARCOVER for \mathcal{C} -inputs G in time $\mathcal{O}^*(2^{e(H)/2})$, where $e(H)$ is the number of edges of H .*

Since the assumptions (P1) to (P3) are satisfied for planar graphs (Lemma 6.2), we get Theorem 1.1. Notice that if the input graph G is 3-connected, using our assumptions the REGULARCOVER problem can be trivially solved, by enumerating all its regular quotients and testing graph isomorphism with H . Babai [5] proved that to solve graph isomorphism, it is sufficient to solve graph isomorphism for 3-connected graphs. We wanted to generalize this result to regular covers, but handling 2-cuts is very complicated and we need the assumptions (P2) and (P3).

We process the graph G by a series of *reductions*, replacing parts of the graph by edges, essentially forgetting details of the graph. We end-up with a primitive graph which is either 3-connected, or very simple (a cycle or K_2). This very natural idea of reductions was first introduced in a seminal paper of Trakhtenbrot [56] and further extended in [57, 37, 19, 59, 5]. The main difference is that these papers apply the reduction only to 2-connected graphs, but in [26], we also reduce parts separated by 1-cuts. The reason is that a regular quotient of a 2-connected graph might be only 1-connected, see Sections 3.1 and 4.1. Also, we prove in [26] that no essential information of semiregular actions is lost during reductions.

In [26], we describe how regular covering behaves with respect to vertex 1-cuts and 2-cuts. Concerning 1-cuts, regular covering behaves non-trivially only on the central block of G , so they are easy to deal with. But regular covering can behave highly complex on 2-cuts. In this paper, we build an algorithm based on these structural results of [26]. When the reductions reach a 3-connected graph, the natural next step is to compute all its quotients; there are polynomially many of them according to (P2).

What remains is the most difficult part: To test for each quotient whether it corresponds to H after unrolling the reductions which is called *expanding*. The issue is that there may be exponentially many different ways to expand the graph, all described in [26]. Therefore, we have to test in a clever way whether it is possible to reach H . Our algorithm consists of several subroutines, most of which we can perform in polynomial time. Only one subroutine (finding a certain “generalized matching”) we have not been able to solve in polynomial time.

This slow subroutine can be avoided in some cases:

COROLLARY 1.4. *If G is a 3-connected graph, if H is a 2-connected graph, or if $k = |G|/|H|$ is odd, then the meta-algorithm of Theorem 1.3 can be modified to run in polynomial time.*

COROLLARY 1.5. *Let \mathcal{C} be a class of graphs satisfying (P1), (P2), and (P3*). There exists an algorithm listing for \mathcal{C} -inputs G all their regular quotients, with a polynomial-time delay.*

Theorem 1.2 implies that to solve the REGULARCOVER problem in general, one has to work with both graphs G and H from the beginning. Our algorithm starts only with G and tries to match its quotients to H only in the end.

Outline. In Section 2, we introduce the formal notation used in this paper. In Section 3, we state key structural properties of atoms, reductions and expansions from [26]. In Section 4, we use them to design the meta-algorithm of Theorem 1.3. In Section 5, we describe more details concerning the only slow subroutine of the meta-algorithm. Finally, in Section 6 we show that the class of planar graphs satisfies (P1) to (P3), thus proving Theorem 1.1. In Conclusions, we describe open problems and possible extensions of our results.

2. Definitions and Preliminaries. In this paper, we work with an extended model of graph which is formally described in [26]. A multigraph G is a pair $(V(G), E(G))$ where $V(G)$ is a set of vertices and $E(G)$ is a multiset of edges. We denote $|V(G)|$ by $v(G) = |G|$ and $|E(G)|$ by $e(G)$. The graph can possibly

contain parallel edges and loops, and each loop at u is incident twice with the vertex u . Each edge $e = uv$ gives rise to two half-edges, one attached to u and the other to v . We denote by $\mathbf{H}(G)$ the collection of all half-edges. We denote $|\mathbf{H}(G)|$ by $\mathbf{h}(G)$ and clearly $\mathbf{h}(G) = 2e(G)$. As quotients, we sometime obtain graphs containing (*standalone*) *half-edges* (missing the opposite half-edges).¹ Also, in the reductions, we obtain *pendant edges*, each consisting of two half-edges, one attached to some vertex u , the other attached to no vertex.

Unless the graph is K_2 , we remove all vertices of degree 1 while keeping both half-edges. Assuming that the original graph contains no pendant edges, this removal does not change the automorphism group and existence of regular covering projections from G to H (when the removal is applied on both G and H). A pendant edge attached to v is called a *single pendant edge* if it is the only pendant edge attached to v . Most graphs in this paper are assumed to be connected.

We consider graphs with colored edges and also with three different edge types (directed edges, undirected edges and a special type called halvable edges). It might seem strange to consider such general objects. But when we apply reductions, we replace parts of the graph by edges and the colors encode isomorphism classes of replaced parts. This allows the algorithm to work with smaller reduced graphs while preserving important parts of the structure of the original large graph. So even if the input graphs G and H are simple, more complicated multigraphs are naturally constructed.

We denote groups by capital Greek letters as for instance Γ . We use \mathbb{S}_n , \mathbb{C}_n , \mathbb{D}_n and \mathbb{A}_n to denote symmetric groups, cyclic groups, dihedral groups and alternating groups, respectively.

2.1. Automorphisms and Groups. We state the definitions in a very general setting of multigraphs and half-edges. An *automorphism* π is fully described by a permutation $\pi_h : \mathbf{H}(G) \rightarrow \mathbf{H}(G)$ preserving edges and incidences between half-edges and vertices. Thus, π_h induces two permutations $\pi_v : \mathbf{V}(G) \rightarrow \mathbf{V}(G)$ and $\pi_e : \mathbf{E}(G) \rightarrow \mathbf{E}(G)$ connected together by the very natural property $\pi_e(uv) = \pi_v(u)\pi_v(v)$ for every $uv \in \mathbf{E}(G)$. In most of situations, we omit subscripts and simply use $\pi(u)$ or $\pi(uv)$. In addition, we require that an automorphism preserves colors, edge types and orientation of directed edges.

Automorphism Groups. Let $\text{Aut}(G)$ be the group of all automorphisms of G . The *orbit* $[v]$ of a vertex $v \in \mathbf{V}(G)$ in the action of $\Gamma \leq \text{Aut}(G)$ is the set of all vertices $\{\pi(v) \mid \pi \in \Gamma \leq \text{Aut}(G)\}$, and the orbit $[e]$ of an edge $e \in \mathbf{E}(G)$ is defined similarly as $\{\pi(e) \mid \pi \in \Gamma \leq \text{Aut}(G)\}$. The *stabilizer* Γ_x of x is the subgroup of all automorphisms which fix x . An action is called *semiregular* if it has no non-trivial (i.e., non-identity) stabilizers of both vertices and half-edges. Further, we require the stabilizer of an edge in a semiregular action to be trivial, unless it is a halvable edge, when it may contain an involution transposing the two half-edges. We say that a group is *semiregular* if the associated action is semiregular. More information on permutation groups can be found in [54].

2.2. Coverings. A graph G *covers* a graph H (or G is a *cover* of H) if there exists a locally bijective homomorphism p called a *covering projection*. A homomorphism p from G to H is given by a mapping $p_h : \mathbf{H}(G) \rightarrow \mathbf{H}(H)$ preserving edges and incidences between half-edges and vertices. It induces two mappings $p_v : \mathbf{V}(G) \rightarrow \mathbf{V}(H)$ and $p_e : \mathbf{E}(G) \rightarrow \mathbf{E}(H)$ such that $p_e(uv) = p_v(u)p_v(v)$ for every $uv \in \mathbf{E}(G)$. The property to be local bijective states that for every vertex $u \in \mathbf{V}(G)$ the mapping p_h restricted to the half-edges incident with u is a bijection. Figure 2.1 contains two examples of graph covers. We mostly omit subscripts and just write $p(u)$ or $p(e)$.

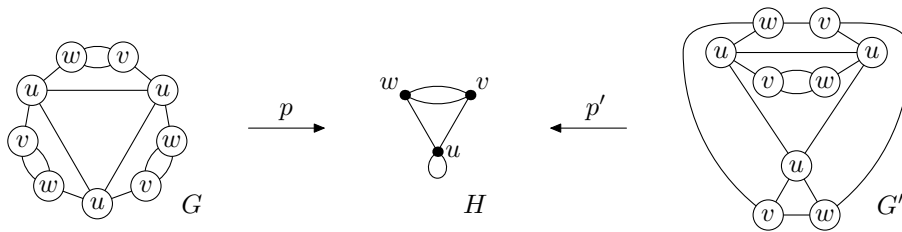


FIG. 2.1. Two covers of H . The projections p_v and p'_v are written inside of the vertices, and the projections p_e and p'_e are omitted. Notice that each loop is realized by having two neighbors labeled the same, and parallel edges are realized by having multiple neighbors labeled the same. Also covering projections preserve degrees.

¹Half-edges are sometimes also called darts or arcs while half-edges with free-ends are called semiedges.

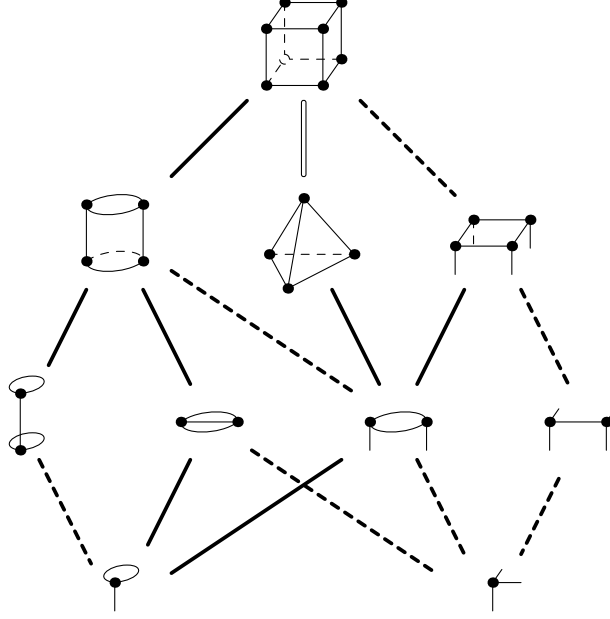


FIG. 2.2. The Hasse diagram of all quotients of the cube graph depicted in a geometric way. When semiregular actions fix edges, the quotients contain half-edges. The quotients connected by bold edges are obtained by 180 degree rotations. The quotients connected by dashed edges are obtained by reflections. The tetrahedron is obtained by the antipodal symmetry of the cube, and its quotient is obtained by a 180 degree rotation with the axis going through the centers of two non-incident edges of the tetrahedron.

A *fiber* over a vertex $v \in V(H)$ is the set $p^{-1}(v)$, i.e., the set of all vertices $V(G)$ that are mapped to v , and similarly for fibers over half-edges. From the standard assumption that both G and H are connected, it follows that all fibers of p are of the same size. In other words, $|G| = k|H|$ for some $k \in \mathbb{N}$, which is the size of each fiber, and we say that G is a k -fold cover of H .

Regular Coverings. We are going to consider coverings which are highly symmetrical, called *regular coverings*. For example, in Fig. 2.1, the covering p is more symmetric than p' . Let Γ be a semiregular subgroup of $\text{Aut}(G)$. It defines a graph G/Γ called a (*regular*) *quotient* of G as follows: The vertices of G/Γ are the orbits of the action of Γ on $V(G)$, the half-edges of G/Γ are the orbits of Γ on $H(G)$. A vertex-orbit $[v]$ is incident with a half-edge-orbit $[h]$ if and only if the vertices of $[v]$ are incident with the half-edges of $[h]$. (Because the action of Γ is semiregular, each vertex of $[v]$ is incident with exactly one half-edge of $[h]$, so this is well defined.)

We naturally construct $p : G \rightarrow G/\Gamma$ by mapping the vertices to its vertex-orbits and half-edges to its half-edge-orbits, and it is a $|\Gamma|$ -fold regular covering. Concerning an edge $e \in E(G)$, it is mapped to an edge of G/Γ if the two half-edges belong to different half-edge-orbits of Γ . If both half-edges belong to the same half-edge-orbits, it corresponds to a standalone half-edge of G/Γ .

For the graphs G and H of Fig. 2.1, we get $H \cong G/\Gamma$ for $\Gamma \cong \mathbb{C}_3$ which “rotates the outer cycle by step three”, while p' is not a regular covering. As a further example, Fig. 2.2 geometrically depicts all regular quotients of the cube graph.

2.3. Complexity of Regular Graph Coverings. We establish fundamental complexity properties of regular covering. Our goal is to highlight similarities with the graph isomorphism problem.

Belonging to NP. The H -COVER problem clearly belongs to NP since one can just test in polynomial time whether a given mapping is a locally bijective homomorphism. Not so obviously, the same holds for REGULARCOVER:

LEMMA 2.1. *The REGULARCOVER problem belongs to NP.*

Proof. By definition, G regularly covers H if and only if there exists a semiregular subgroup Γ of $\text{Aut}(G)$ such that $G/\Gamma \cong H$. As a certificate, we give k permutations, one for each element of Γ , and an isomorphism between G/Γ and H . We check that these permutations define a group Γ acting semiregularly on G . The

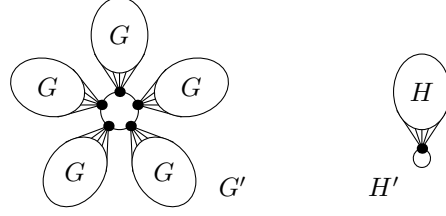


FIG. 2.3. The graph G' is constructed by k copies of G with attached universal vertices connected into a cycle while H' is constructed by attaching a universal vertex with a loop to H .

given isomorphism allows to check that the constructed G/Γ is isomorphic to H . Clearly, this certificate is polynomially large and can be verified in polynomial time. \square

One can prove even a stronger result:

LEMMA 2.2. *For a mapping $p : G \rightarrow H$, we can test whether it is a regular covering projection in polynomial time.*

Proof. Testing whether p is a covering projection can clearly be done in polynomial time. It remains to test regularity. Choose an arbitrary spanning tree T of H . Since p is a covering, then $p^{-1}(T)$ is a disjoint union of k isomorphic copies T_1, \dots, T_k of T . We number the vertices of the fibers according to the spanning trees, i.e., $p^{-1}(v) = \{v_1, \dots, v_k\}$ such that $v_i \in T_i$. This induces a numbering of the half-edges of each fiber over a half-edge of $H(H)$, following the incidences between half-edges and vertices. For every half-edge $h \notin H(T)$, we define the permutation σ_h of $\{1, \dots, k\}$ taking i to j if there is a half-edge $h' \in p^{-1}(h)$ incident with a vertex of T_i and paired with a half-edge incident with a vertex of T_j .

Let Θ be the group generated by all σ_h , where $h \notin H(T)$. We assume that G is connected. By Orbit-Stabilizer Theorem, we have $|\Theta| = |\Theta_v| \cdot |[v]|$, and from the connectivity, it follows that $[v] = k$. Therefore, the action of Θ is regular if and only if $|\Theta| = k$ which can be checked in polynomial time. \square

The constructed permutations σ_h associated with p are known in the literature [31] as *permutation voltage assignments* associated with p .

GI-hardness. When $k = |G|/|H| = 1$, the problem REGULARCOVER exactly corresponds to GRAPHISO. Let GI be the class of decision problems polynomial-time reducible to GRAPHISO. Bodlaender [12] proved the following for general covers (and his reduction works for regular covers as well):

LEMMA 2.3. *For every fixed k , the k -FOLDCOVER and k -FOLDREGULARCOVER problems are GI-hard.*

Proof. For input graphs G and H of the graph isomorphism problem, we construct the graphs G' and H' depicted in Fig. 2.3. The reduction works since the universal vertices in G' must be mapped to the universal vertex in H' (since covering projection preserves degrees). Therefore, G' (regularly) covers H' if and only if $G \cong H$. \square

3. Atoms, Reduction and Expansion. In this section, we state definitions and structural results from [26], describing behaviour of regular graph covers with respect to 1-cuts and 2-cuts in G . We also prove new results concerning computational complexity of these techniques. In Section 3.1, we introduce block-trees and describe behaviour of regular covering with respect to 1-cuts. In Section 3.2, we introduce atoms which are inclusion-minimal parts of G with respect to 1-cuts and 2-cuts. In Section 3.3, we describe the reduction which replaces atoms by colored edges, preserving the essential structure of G . In Section 3.4, we consider quotients of reduced graphs and revert the reductions in them by expansions.

3.1. Block-trees and Central Blocks. The *block-tree* T of G is defined as follows. Consider all articulations in G and all maximal 2-connected subgraphs which we call *blocks* (with bridge-edges and pendant edges also counted as blocks). The block-tree T is the incidence graph between the articulations and the blocks. For an example, see Fig. 3.1.

The Central Block. Recall that for a tree, its *center* is either the central vertex or the central pair of vertices of a longest path, depending on the parity of its length. Every automorphism of a tree preserves its center.

LEMMA 3.1 ([26], Lemma 2.1). *If G has a non-trivial semiregular automorphism, then G has a central block.*

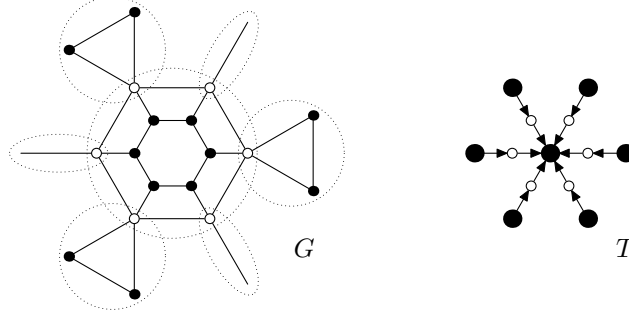


FIG. 3.1. On the left, an example graph G with denoted blocks. On the right, the corresponding block-tree T is depicted, rooted at the central block. The white vertices correspond to the articulations and the big black vertices correspond to the blocks.

In the following, we shall assume that T contains a central block C . We orient the edges of the block-tree T towards the central block; so the block-tree becomes rooted. A (*rooted*) *subtree* of the block-tree is defined by any vertex different from the central block acting as *root* and by all its descendants. Let u be an articulation contained in C . By T_u we denote the subtree of T defined by u and all its predecessors, and let G_u be the graph induced by all vertices of the blocks of T_u . All semiregular subgroups of $\text{Aut}(G)$ act non-trivially and faithfully only on C :

LEMMA 3.2 ([26], Lemma 2.2). *Let Γ be a semiregular subgroup of $\text{Aut}(G)$. If u and v are two articulations of the central block and of the same orbit of Γ , then $G_u \cong G_v$. Moreover there is a unique $\pi \in \Gamma$ which maps G_u to G_v .*

In the language of quotients, it means that G/Γ consists of C/Γ together with the graphs G_u attached to C/Γ , one for each orbit of Γ .

Why Not Just 2-connected Graphs? Since the behaviour of regular covering with respect to 1-cuts in G is very simple, a natural question follows: why do we not restrict ourselves to 2-connected graphs G ? For instance when solving graph isomorphism, it is sufficient to solve graph isomorphism of 2-connected graphs and use it to find isomorphism of block-trees.

This is not possible for regular covering testing. The issue is that the quotient C/Γ might not be 2-connected, so it may consist of many blocks and it is not easy to locate it in H . When H contains a subtree of blocks isomorphic to G_u , it may correspond to G_u , or it may correspond to a quotient of a subgraph of C/Γ , together with some other G_v attached. We use dynamic programming to deal with this in the meta-algorithm, see Section 4.1. Therefore, we have to define 3-connected reduction for 1-cuts in G as well, unlike in [56, 37, 19, 59, 5].

3.2. Atoms. Suppose that B is a block of G , in particular B is 2-connected. Two vertices u and v form a *2-cut* $U = \{u, v\}$ if $B \setminus U$ is disconnected. We say that a 2-cut U is *non-trivial* if $\deg(u) \geq 3$ and $\deg(v) \geq 3$ in B .

Atoms are inclusion-minimal subgraphs with respect to 1-cuts and 2-cuts in G . We first define a set \mathcal{P} of subgraphs of G called *parts* which are candidates for atoms:

- A *block part* is a subgraph non-isomorphic to a pendant edge induced by the blocks of a subtree of the block-tree.
- A *proper part* is a subgraph S of G defined by a non-trivial 2-cut U of a block B . The subgraph S consists of a connected component K of $G \setminus U$ together with u and v and all edges between $\{u, v\}$ and K . In addition, we require that S does not contain the central block; so it only contains some blocks of the subtree of the block-tree rooted at B .
- A *dipole part* is any dipole defined as follows. Let u and v be two distinct vertices of degree at least three joined by at least two parallel edges. Then the subgraph induced by u and v is called a *dipole*.

The inclusion-minimal elements of \mathcal{P} are called *atoms*. We distinguish *block atoms*, *proper atoms* and *dipoles* according to the type of the defining part. Block atoms are either stars of pendant edges called *star block atoms*, or pendant blocks possibly with single pendant edges attached to them called *non-star block atoms*. Each proper atom is a subgraph of a block, together with some single pendant edges attached to it. A dipole part is by definition always inclusion-minimal, and therefore it is an atom. For an example, see Fig. 3.2.

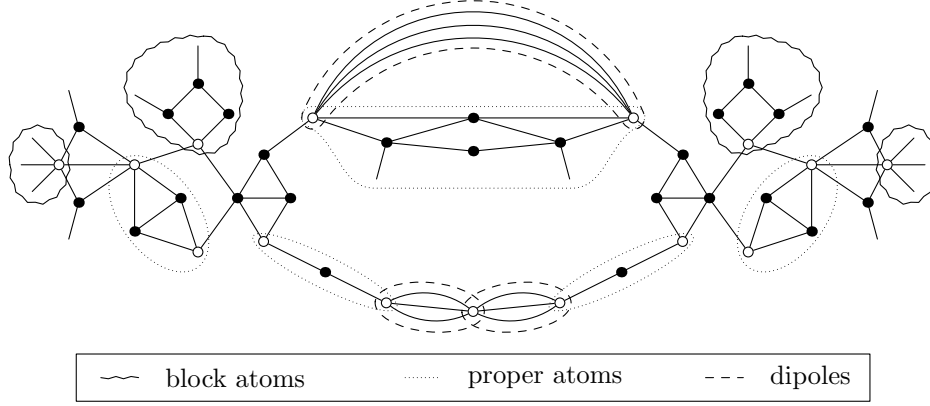


FIG. 3.2. An example of a graph with denoted atoms. The white vertices belong to the boundary of some atom, possibly several of them.

We use the topological notation to denote the *boundary* ∂A and the *interior* $\overset{\circ}{A}$ of an atom A . If A is a dipole, we set $\partial A = V(A)$. If A is a proper or block atom, we put ∂A equal to the set of vertices of A which are incident with an edge not contained in A . For the interior, we use the standard topological definition $\overset{\circ}{A} = A \setminus \partial A$ where we only remove the vertices ∂A , the edges adjacent to ∂A are kept in $\overset{\circ}{A}$. Single pendant edges of A are always attached to $\overset{\circ}{A}$.

Note that $|\partial A| = 1$ for a block atom A , and $|\partial A| = 2$ for a proper atom or dipole A . The interior of a dipole is a set of free edges. For a proper atom A , the vertices of ∂A are exactly the vertices $\{u, v\}$ of the non-trivial 2-cut used in the definition of proper parts, and they are never adjacent in A .

LEMMA 3.3 ([26], Lemma 3.3). *Let A and A' be two different atoms. Then $A \cap A' = \partial A \cap \partial A'$.*

LEMMA 3.4 ([26], Lemma 3.8). *Let A be an atom and let $\pi \in \text{Aut}(G)$.*

- (a) *The image $\pi(A)$ is an atom isomorphic to A . Moreover $\pi(\partial A) = \partial \pi(A)$ and $\pi(\overset{\circ}{A}) = \overset{\circ}{\pi(A)}$, where $\overset{\circ}{\pi(A)}$ denotes the interior of $\pi(A)$.*
- (b) *If $\pi(A) \neq A$, then $\pi(\overset{\circ}{A}) \cap \overset{\circ}{A} = \emptyset$.*
- (c) *If $\pi(A) \neq A$, then $\pi(A) \cap A = \partial A \cap \partial \pi(A)$.*

Primitive Graphs. A graph is called *primitive* if it contains no atoms. The following lemma characterizing primitive graphs can be alternatively obtained from the well-known theorem by Trakhtenbrot [56], see Fig. 3.3 for examples.²

LEMMA 3.5 ([26], Lemma 3.4). *Let G be a primitive graph. If G has a central block, then it is a 3-connected graph, a cycle C_n for $n \geq 2$, or K_2 , or can be obtained from the aforementioned graphs by attaching single pendant edges to at least two vertices. If G has a central articulation, then it is K_1 , possible with a single pendant edge attached.*

LEMMA 3.6. *If primitive graphs G and G' belong to \mathcal{C} satisfying $(P3^*)$, then we can test $G \cong G'$ in polynomial time.*

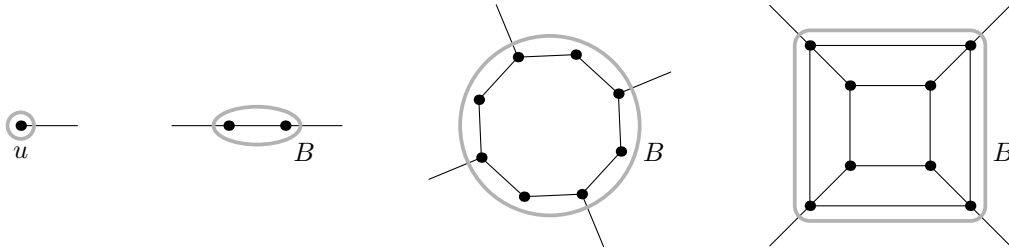


FIG. 3.3. A primitive graph with a central block is either K_2 , C_n , or a 3-connected graph, in all three cases with possible single pendant edges attached to it.

²We consider K_1 with an attached single pendant edge as a graph with a central articulation.

Proof. In both graphs, we replace single pendant edges with colored vertices. If G and G' are K_1 , or K_2 , the problem is trivial. If they are cycles, we use the standard cycle isomorphism algorithms. If they are 3-connected, we test $G \cong G'$ using (P3*). \square

Structure of Atoms. We call a graph *essentially 3-connected* if it is a 3-connected graph with possibly single pendant edges attached to it. Similarly, a graph is called *essentially a cycle* if it is a cycle with possibly single pendant edges attached to it. The structure of dipoles and star block atoms is clear. Non-star block and proper atoms are either almost 3-connected, or very simple:

LEMMA 3.7 ([26], Lemma 3.5). *Every non-star block atom A is either K_2 with an attached single pendant edge, essentially a cycle, or essentially 3-connected.*

Let A be a proper atom with $\partial A = \{u, v\}$. We define the *extended proper atom* A^+ as A with the additional edge uv . Notice that the property (P1) ensures that A^+ belongs to the class \mathcal{C} .

LEMMA 3.8 ([26], Lemma 3.6). *For every proper atoms A , the extended proper atom A^+ is either essentially a cycle, or essentially 3-connected.*

Two atoms A and A' are isomorphic if there exists an isomorphism which maps ∂A to $\partial A'$.

LEMMA 3.9. *For every atoms A and A' of a graph G belonging to \mathcal{C} satisfying (P1) and (P3*), we can test $A \cong A'$ in polynomial time.*

Proof. If both A and A' are dipoles and star block atoms, we can test $A \cong A'$ trivially in polynomial time. If they are non-star block atoms, by Lemma 3.7 they are either K_2 with attached single pendant edge, or essentially a cycle, or essentially 3-connected. The first two possibilities can be solved trivially, so we assume that A and A' are essentially 3-connected. Let B and B' be the 3-connected graph created from A and A' by removing pendant edges, where existence of pendant edges is coded by colors of $V(B)$ and $V(B')$, and we further color ∂B and $\partial B'$ by a special color. We have $A \cong A'$ if and only if there exists a color-preserving isomorphism between B and B' which can be tested using (P3*). When A and A' are proper atoms, we proceed similarly on extended proper atoms, using Lemma 3.8. \square

Symmetry Types of Atoms. We distinguish three symmetry types of atoms, and in reductions, we replace atoms by edges carrying their types. Therefore we work with multigraphs with three edge types: *halvable edges*, *undirected edges* and *directed edges*. We consider only the automorphisms which preserve these edge types and of course the orientation of directed edges. For an atom A , we denote by $\text{Aut}(A)$ the setwise stabilizer of ∂A .

Let A be a proper atom or dipole with $\partial A = \{u, v\}$. We distinguish the following three symmetry types, depicted in Fig. 3.4:

- *The halvable atom.* There exists a semiregular involutory automorphism $\tau \in \text{Aut}(A)$ which exchanges u and v . More precisely, the automorphism τ fixes no vertices and no directed and undirected edges, but some halvable edges may be fixed.
- *The symmetric atom.* The atom is not halvable, but there exists an automorphism in $\text{Aut}(A)$ which exchanges u and v .
- *The asymmetric atom.* The atom is neither halvable, nor symmetric.

If A is a block atom, then it is by definition symmetric.

LEMMA 3.10. *For a dipole A , we can determine its symmetry type in polynomial time.*

Proof. The type depends only on the quantity of distinguished types of the parallel edges. We have directed edges from u to v , directed edges from v to u , undirected edges and halvable edges. We call a dipole *balanced* if the number of directed edges in the both directions is the same. The dipole is halvable if and

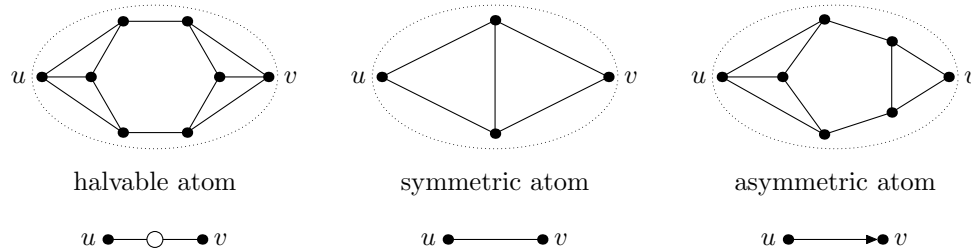


FIG. 3.4. *The three types of atoms and the corresponding edge types which we use in the reduction. We denote halvable edges by small circles in the middle.*

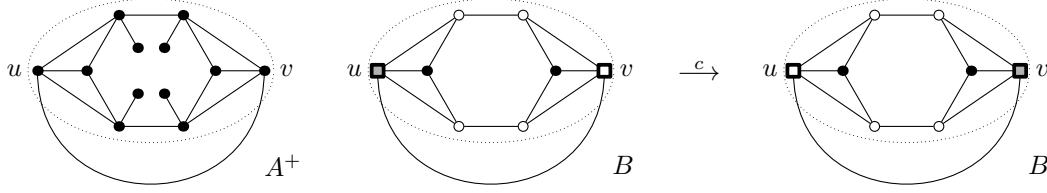


FIG. 3.5. For the depicted atom A , we test using $(P3^*)$ whether $B \xrightarrow{c} B$. In this case yes, so A is either symmetric, or halvable.

only if it is balanced and has an even number of undirected edges. The dipole is symmetric if and only if it is balanced and has an odd number of undirected edges. The dipole is asymmetric if and only if it is unbalanced. This clearly can be tested in polynomial time. \square

LEMMA 3.11. *For a proper atom A of \mathcal{C} satisfying $(P1)$, $(P2)$, and $(P3^*)$, we can determine its symmetry type in polynomial time.*

Proof. Let $\partial A = \{u, v\}$. By Lemma 3.8, A^+ is either essentially a cycle (which is easy to deal with), or an essentially 3-connected graph. Let B be the 3-connected graph created from A^+ by removing pendant edges, where existence of pendant edges is coded by colors of $\mathbf{V}(B)$. By $(P1)$, both A^+ and B belong to \mathcal{C} . We apply $(P3^*)$ on two copies of B . In one copy, we color u by a special color, and v by another special color. In the other copy, we swap the colors of u and v . Using $(P3^*)$, we check whether there exists a color-preserving automorphism which exchanges u and v ; see Fig. 3.5. If not, then A is asymmetric. If yes, we check whether A is symmetric or halvable.

Using $(P2)$, we generate polynomially many semiregular involutions of order two acting on B . For each semiregular involution, we check whether it transposes u to v , and whether it preserves the colors of $\mathbf{V}(B)$ coding pendant edges. If such a semiregular involution exists, then A is halvable, otherwise it is just symmetric. \square

Regular Projections and Quotients of Atoms. Let Γ be a semiregular subgroup of $\text{Aut}(G)$, which defines a regular covering projection $p : G \rightarrow G/\Gamma$. For a proper atom or a dipole A with $\partial A = \{u, v\}$, we get three possible *types of projection* $p|_A$; see Fig. 3.6:

- *An edge-projection.* The atom A is preserved in G/Γ , meaning $p(A) \cong A$. Notice that $p(A)$ may just be a subgraph of G/Γ , not induced. For instance, it can happen that $p(u)p(v) \in \mathbf{E}(G/\Gamma)$ while $uv \notin \mathbf{E}(G)$.
- *A loop-projection.* The interior \mathring{A} is preserved and the vertices u and v are identified, i.e., $p(\mathring{A}) \cong \mathring{A}$ and $p(u) = p(v)$.
- *A half-projection.* There exists an involutory permutation π in Γ which exchanges u and v and preserves A . The projection $p(A)$ is a halved atom A . This can happen only when A is a halvable atom. In particular, the covering projection p is a $2k$ -fold covering.

LEMMA 3.12 ([26], Lemma 3.9). *For an atom A and a regular covering projection p , we have $p|_A$ either an edge-projection, a loop-projection, or a half-projection. Moreover, for a block atom we have exclusively an edge-projection.*

So we get three types of quotients $p(A)$ of A . For an edge-projection, we call this quotient an *edge-quotient*, for a loop-projection, we call it a *loop-quotient*, and for a half-projection, we call it a *half-quotient*. The following lemma allows to say “the” edge- and “the” loop-quotient of an atom.

LEMMA 3.13 ([26], Lemma 3.9). *For every atom A , there is the unique edge-quotient and the unique loop-quotient up to isomorphism.*

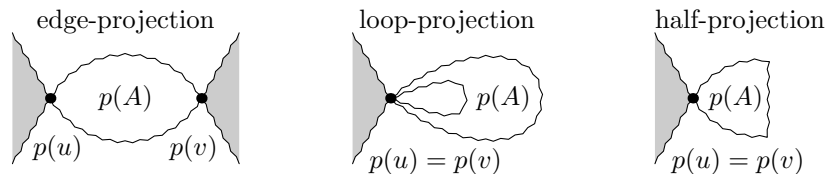


FIG. 3.6. How can $p(A)$ look in G/Γ , for three types of projection.

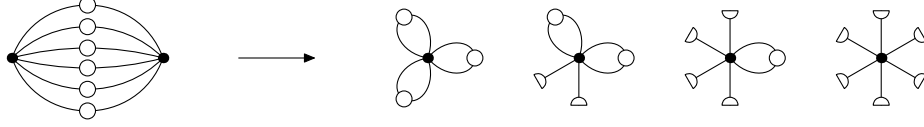


FIG. 3.7. Assuming that quotients can contain half-edges, the depicted dipole has four non-isomorphic half-quotients.

For half-quotients, this uniqueness does not hold. First, an atom A with $\partial A = \{u, v\}$ has to be halvable to admit a half-quotient. Then each half-quotient is determined by an involutory automorphism τ exchanging u and v ; here τ is the restriction of π from the definition of a half-projection. First, several different automorphisms τ may define equivalent covering projections p of A , so $p(A)$ is an isomorphic half-quotient. On the other hand, different automorphisms τ may define non-equivalent covering projections p of A , so they give non-isomorphic half-quotients $p(A)$; see Fig. 3.7. For a proper atom, we can bound the number of non-isomorphic half-quotients by the number of different semiregular involutions of 3-connected graphs.

LEMMA 3.14. *Let A be a proper atom of \mathcal{C} satisfying (P1) and (P2). Then there are polynomially many non-isomorphic half-quotients of A which can be computed in polynomial time.*

Proof. By Lemma 3.8, A^+ is either essentially a cycle (where it holds trivially), or it is an essentially 3-connected graph. We construct B^+ from A^+ by replacing pendant edges with colored vertices, by (P1) both A^+ and B^+ belong to \mathcal{C} . According to (P2), the number of different semiregular subgroups of order two is polynomial in the size of B^+ . Each half-quotient is defined by one of these semiregular involutions which fixes the edge uv , transposes u and v , and preserves colors. \square

3.3. Reduction. The reduction produces a *reduction series* of graphs $G = G_0, \dots, G_r$. It produces graphs with colored edges and with three edge types: halvable, undirected and directed. We note that the results built in Section 3.2 transfers to colored graphs and colored atoms without any problems.

To construct G_{i+1} from G_i , we find the collection of all atoms \mathcal{A} of G_i , together with isomorphism classes such that A and A' belong to the same class if and only if $A \cong A'$. To each isomorphism class, we assign one new color not yet used in the graphs G_0, \dots, G_i . We replace the atoms \mathcal{A} in G_i by edges of colors of the corresponding isomorphism classes as follows.

For each block atom A with $\partial A = \{u\}$, we replace it by a pendant edge of some color based at u . For each proper atom or dipole A with $\partial A = \{u, v\}$, we replace it by a new edge uv which is halvable/undirected/directed when A is halvable/symmetric/asymmetric, respectively. Naturally, for each isomorphism class of asymmetric atom, we consistently choose an arbitrary orientation of the directed edges replacing these atoms. For an example of the reduction, see Fig. 3.8. By Lemma 3.3, the replaced the interiors of the atoms of \mathcal{A} are pairwise disjoint, so the reduction is well defined.

The reduction series stops in the step r when G_r is a primitive graph. For every graph G , the reduction series corresponds to the *reduction tree* which is a rooted tree defined as follows. The root is the primitive graph G_r , and the other nodes are the atoms obtained during the reductions. If a node contains a colored edge, it has the corresponding atom as a child. Therefore, the leaves are the atoms of G_0 , after removing

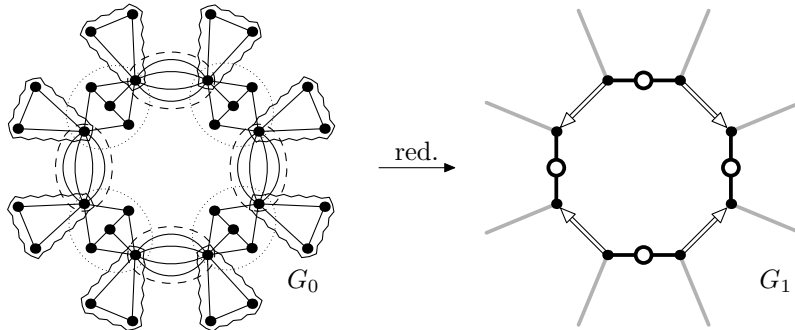


FIG. 3.8. On the left, we have a graph G_0 with three isomorphism classes of atoms. The dipoles are halvable, the block atoms are symmetric and the proper atoms are asymmetric. We reduce G_0 to G_1 which is an eight cycle with single pendant edges, with four black halvable edges replacing the dipoles, eight gray undirected edges replacing the block atoms, and four white directed edges replacing the proper atoms. The reduction series ends with G_1 since it is primitive.

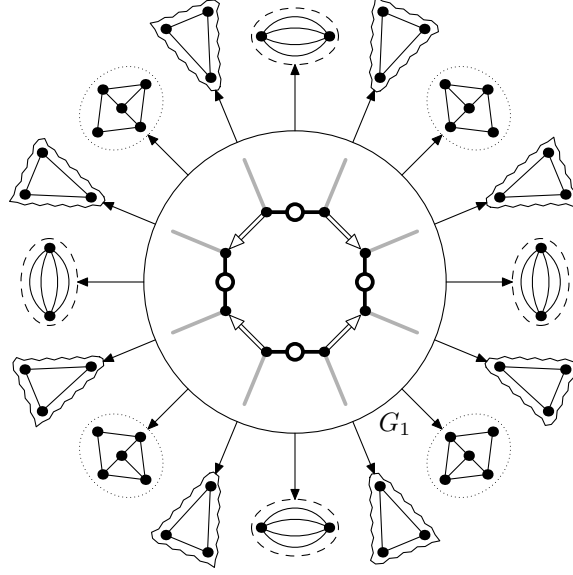


FIG. 3.9. The reduction tree for the reduction series in Fig. 3.8. The root is the primitive graph G_1 and each leaf corresponds to one atom of G_0 .

them, the new leaves are the atoms of G_1 , and so on. For an example, see Fig. 3.9. It is proved in [26, 42] that the reduction series and the reduction tree captures $\text{Aut}(G)$.

LEMMA 3.15. *If a graph G belongs to \mathcal{C} satisfying (P1), (P2) and (P3*), then the reductions series $G = G_0, \dots, G_r$ and the reduction tree can be computed in polynomial time.*

Proof. To compute G_{i+1} from G_i , we find all atoms \mathcal{A} in G_i and \mathcal{A}' in G'_i , compute their isomorphism classes by Lemma 3.9 and assign new colors to them. By Lemmas 3.10 and 3.11, we compute symmetry types of these atoms. We end up with a primitive graph G_r containing the atoms. The reduction tree can be easily constructed and the algorithm runs in polynomial time. \square

The following is the approach invented by Babai [5]:

LEMMA 3.16 ([5]). *If graphs G and G' belong to \mathcal{C} satisfying (P1) and (P3*), we can test $G \cong G'$ in polynomial time.*

Proof. Using Lemma 3.15, we simultaneously apply reduction series on both G and G' in polynomial time, using identical colors for isomorphic atoms in G_i and G'_i . (We do not need to distinguish halvable and symmetric atoms, so (P2) is not needed.) We end up with two primitive graphs G_r and G'_r and we test their isomorphism using Lemma 3.6. Alternatively, we can compute both reduction trees and apply the standard tree isomorphism of graph-labeled trees. \square

In general, the reduction series does not have to preserve the central block, and the atoms \mathcal{A} of G_0, \dots, G_{r-1} has to be defined with respect to one chosen block which is preserved. On the other hand, by Lemma 3.1, if the REGULARCOVER problem is non-trivial, then G contains the central block which is

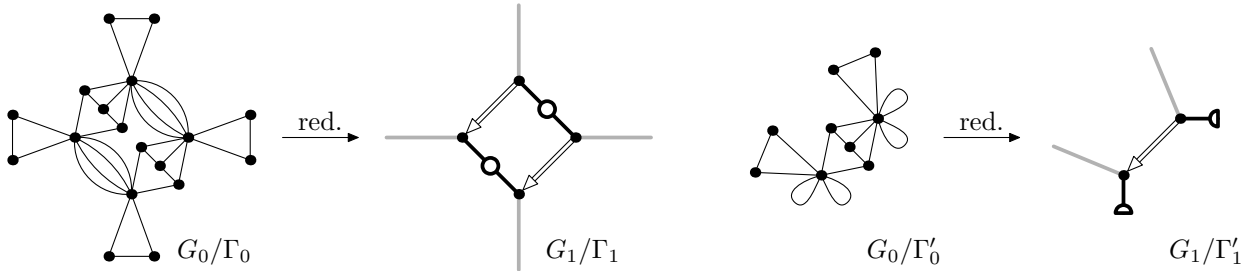


FIG. 3.10. An example of two quotients G_0/Γ_0 and G_0/Γ'_0 of the graph G_0 from Fig. 3.8 with the corresponding quotients of the reduced graph G_1 .

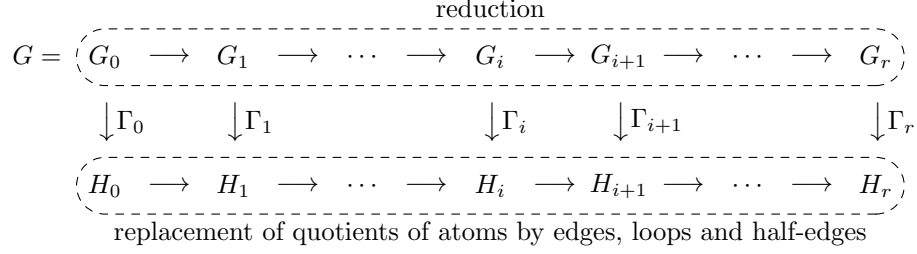


FIG. 3.11. The graph H_{i+1} is constructed from H_i by replacing the projections of atoms in H_i by the corresponding projections of the edges replacing the atoms.

preserved by the reduction series:

LEMMA 3.17 ([26], Lemma 4.1). *Let G admit a non-trivial semiregular automorphism π . Then each G_{i+1} has a central block which is obtained from the central block of G_i by replacing its atoms by colored edges.*

Quotient Reduction. Let G_0, \dots, G_r be the reduction series of G and let Γ_0 be a semiregular subgroup of $\text{Aut}(G_0)$. By Lemma 3.4, we argue that Γ_0 uniquely determines semiregular subgroups $\Gamma_1, \dots, \Gamma_r$ of $\text{Aut}(G_1), \dots, \text{Aut}(G_r)$. Each element $\pi \in \Gamma_i$ somehow permutes atoms \mathcal{A} of G_i and somehow permutes the rest of G_i . The reduction constructs G_{i+1} from G_i by replacing \mathcal{A} with colored edges. Therefore, $\pi' \in \Gamma_{i+1}$ corresponding to π can be defined in the following way. It permutes the colored edges in the same way as π permutes the respective atoms of \mathcal{A} , while π' is equal to π on the rest of the graph. The mapping $\pi \mapsto \pi'$ defines the *reduction epimorphism* $\Phi_i : \text{Aut}(G_i) \rightarrow \text{Aut}(G_{i+1})$; see [26], Proposition 4.1.

Let $H_i = G_i/\Gamma_i$ be the quotients with preserved colors and types of edges, and let p_i be the corresponding covering projection from G_i to H_i . Recall that H_i can contain edges, loops and half-edges; depending on the action of Γ_i on the half-edges corresponding to the edges of G_i . We investigate relation between $H_i = G_i/\Gamma_i$ and $H_{i+1} = G_{i+1}/\Gamma_{i+1}$.

Let A be an atom of G_i represented by a colored edge e in G_{i+1} . By Lemma 3.12, $p_i|_A$ can have three types of projections. It is easy to see that $p_{i+1}(e)$ corresponds to an edge (for a block atom, to a pendant edge) for the edge-projection, to a loop for the loop-projection and to a half-edge for a half-projection. This explains the names of the quotients $p_i(A)$ as the edge-quotient, the loop-quotient and a half-quotient. See Fig. 3.10 for examples and see the diagram in Fig. 3.11.

3.4. Expansion. We want to understand the *expansion* of quotients, corresponding to the diagram in Fig. 3.12. Suppose that Γ_r is a semiregular subgroup of $\text{Aut}(G_r)$, defining the quotient $H_r = G_r/\Gamma_r$. The expansion constructs a series of semiregular subgroups $\Gamma_r, \dots, \Gamma_0$ defining expanded quotients $H_i = G_i/\Gamma_i$.

Unfortunately, the expansion is non-deterministic which means that the expanded quotients H_{r-1}, \dots, H_0 are not uniquely determined. Recall Lemma 3.12 and Fig. 3.6. The following characterization of all expanded quotients is the main result of [26]:

THEOREM 3.18 ([26], Theorem 1.2). *Let G_{i+1} be a reduction of G_i . Every quotient H_i of G_i can be constructed from some quotient H_{i+1} of G_{i+1} by replacing each edge, loop and half-edge of H_{i+1} by the subgraph corresponding to the edge-, the loop-, or a half-quotient of an atom of G_i , respectively.*

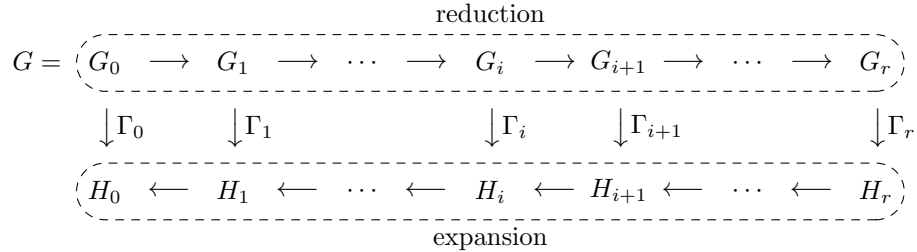


FIG. 3.12. The expansion constructs the graph H_i from H_{i+1} by replacing the edges, loops and half-edges corresponding to quotients of atoms in H_{i+1} by the edge-, the loop- and some choices of half-quotients.

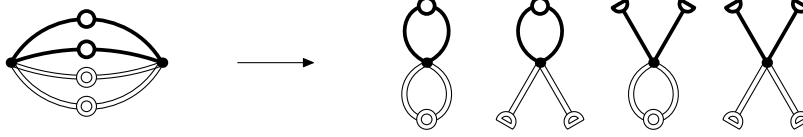


FIG. 3.13. An example of a dipole with four non-isomorphic half-quotients.

By Lemma 3.13, the edge and loop-quotients are uniquely determined. The expansion is non-deterministic since there might be many non-isomorphic half-quotients, leading to different graphs H_i . For instance, suppose that H_{i+1} contains a half-edge corresponding to the dipole from Fig. 3.13. To construct H_i , we replace this half-edge by one of the four possible half-quotients of this dipole.

COROLLARY 3.19 ([26], Corollary 4.8). *If H_{i+1} contains no half-edge, then H_i is uniquely determined. Thus, for an odd order of Γ_r , the quotient H_r uniquely determines H_0 .*

Half-quotients of Dipoles. Dipoles with colored edges may admit exponentially many non-isomorphic half-quotients; see Fig. 3.13. Therefore, if H_{i+1} contains a half-edge corresponding to a half-quotient of a dipole in H_i , the number of non-isomorphic expansions H_i of H_{i+1} can be exponential in the size difference of H_i and H_{i+1} .

LEMMA 3.20 ([26], Lemma 4.9). *Let A be a dipole with colored edges. Then the number of pairwise non-isomorphic half-quotients is bounded by $2^{\lfloor e(A)/2 \rfloor}$ and this bound is achieved.*

For the purpose of Section 4, we describe the structure of all quotients of a dipole. Each is constructed from an involutory semiregular automorphisms τ acting on \dot{A} . If $\partial A = \{u, v\}$, the half-quotient $A/\langle\tau\rangle$ consists of a vertex $p(u)$ with several loops and half-edges attached at u . Since τ preserves the color classes and edge types, it acts independently on each color class and type of edges.

On each color class of the non-halvable edges of A , τ acts as a fixed-point free involution. The undirected edges have to be paired by τ together. Each directed edge has to be paired with a directed edge of the opposite direction and the same color. In the quotient $A/\langle\tau\rangle$, we have no freedom: we get a (directed) loop for each such pair.

On each color class of halvable edges of A , τ acts as an arbitrary semiregular involution. An edge e fixed in τ is mapped into a half-edge of the given color in $A/\langle\tau\rangle$. If τ maps e to $e' \neq e$, then we get a loop in $A/\langle\tau\rangle$. The resulting half-quotient is therefore determined by the numbers h of fixed edges and the numbers ℓ of two-cycles for each color class of halvable edges of size m such that $h + 2\ell = m$.

The Block Structure of Quotients. Last, we describe how the block structure changes during expansions. A block atom A of G_i is always projected by an edge-projection, so it corresponds to a block atom of H_i . Suppose that A is a proper atom or a dipole with $\partial A = \{u, v\}$. For an edge-projection, we get $p(u) \neq p(v)$, and $p(A)$ is isomorphic to an atom in H_i .

For a loop- or half-projection, we get $p(u) = p(v)$ and $p(u)$ is an articulation of H_i . If A is a dipole, then $p(A)$ is a pendant star of half-edges and loops attached to $p(u)$. By Lemma 3.8, if A is a proper atom, then $p(A)$ is either a path ending with a half-edge and with attached single pendant edges (when A^+ is essentially a cycle), or a pendant block with attached single pendant edges and half-edges (when A^+ is essentially 3-connected). (The reason is that the fiber of an articulation in a 2-fold cover is a 2-cut.)

LEMMA 3.21 ([26], Lemma 4.10). *The block structure of H_{i+1} is preserved in H_i , possibly with some new subtrees of blocks attached.*

4. Meta-algorithm. In this section, we establish the meta-algorithm from Theorem 1.3, solving REGULARCOVER for G belonging to \mathcal{C} satisfying (P1) to (P3) in time $\mathcal{O}^*(2^{e(H)/2})$.

Let $k = |G|/|H|$, and we assume that $k \geq 2$. (If k is not an integer, then clearly G does not cover H . If $k = 1$, then it is equivalent to the graph isomorphism problem and we can test it using Lemma 3.16.) The algorithm consists of the following major parts:

1. *Reduction Part:* We construct the reduction series for $G = G_0, \dots, G_r$ terminating with the unique primitive graph G_r . Throughout the reduction the central block is preserved, otherwise according to Lemma 3.17 there exists no semiregular automorphism of G and we output “no”. According to (P1), the reduction preserves the class \mathcal{C} , and also every atom belongs to \mathcal{C} .
2. *Quotient Part:* Using (P2), we construct the list of all subgroups Γ_r of $\text{Aut}(G_r)$ of the order k acting semiregularly on G_r . The number of subgroups in the list is polynomially large by (P2).

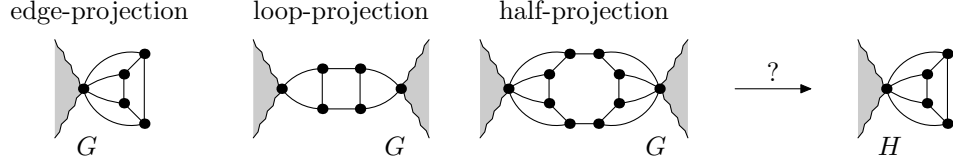


FIG. 4.1. For a pendant block of H , there are three possible preimages in G . It could be a block atom mapped by the edge-projection, or a proper atom mapped by the loop-projection, or another proper atom mapped by a half-projection (where the half-quotient is created by 180° rotation τ).

3. *Expansion Part:* For each Γ_r in the list, we compute $H_r = G_r/\Gamma_r$. We say that a graph H_r is *expandable* if there exists a sequence of extensions repeatedly applying Theorem 3.18 which constructs H_0 isomorphic to H . We test the expandability of H_r using dynamic programming while using (P3). It remains to explain details of Expansion Part, and prove the correctness of the algorithm.

Outline. In Section 4.1, we give an overview of Expansion Part. In Section 4.2, we describe a catalog which stores all atoms and their quotients discovered during reductions. In Section 4.3, we describe reductions with lists, used in expandability testing. Last, in Section 4.4, we conclude with a proof of Theorem 1.3.

4.1. Overview of Testing Expandability. In this section, we explain how to test expandability of H_r . We start by illustrating the fundamental difficulty, for simplicity on pendant blocks. Suppose that H has a pendant block as in Fig. 4.1. From the local information, there is no way to know whether this block corresponds in G to the edge-quotient of a block atom, or to the loop-quotients of some proper atoms, or to half-quotients of some other proper atoms. It can easily happen that the all these atoms appear in G . So without exploiting some additional information from H , there is no way to know what is the preimage of this pendant block.

In our approach, we revert the problem of expandability of H_r by reducing H towards H_r . But since it is not clear which atoms of H correspond to which parts of G , we do not decide it during the reductions, instead we just remember lists of all possibilities. The dynamic programming deals with these lists and computes further lists for larger parts of H . Figure 4.2 illustrates the overview of our algorithm.

Reductions of Quotients and Cores. Notice that H_r might not be primitive; see Fig. 4.3 for an example. It would be difficult to match it to a reduction series in H , so in Step 3, we further reduce H_r to a primitive graph H_s .

We define atoms in the quotient graphs similarly as in Section 3.2 with only one difference. We choose one arbitrary block/articulation called the *core* in H_r ; for instance, we can choose the central block/articulation. The core plays the role of the central block in the definition of parts and atoms. Also, in the definition we consider half-edges and loops as pendant edges, so they do not form block atoms. We proceed with the reductions in H_r further till we obtain a primitive quotient graph H_s , for some $s \geq r$; see Fig. 4.4.

Let H_0, \dots, H_{s-1} be the graphs obtained by an expansion series of H_s using Theorem 3.18.

LEMMA 4.1. *The graph H_s is expandable to H , if and only if H_r is expandable to H .*

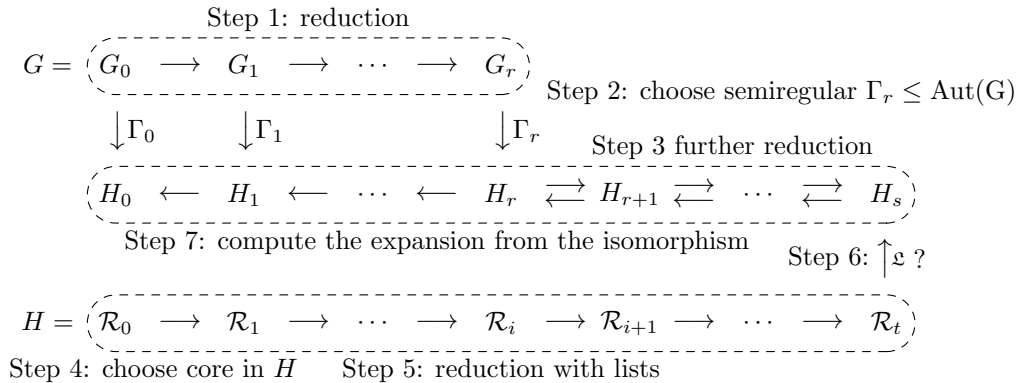


FIG. 4.2. The metaalgorithm proceeds in the following seven steps. We iterate over all possible choices in Steps 2 and 4.

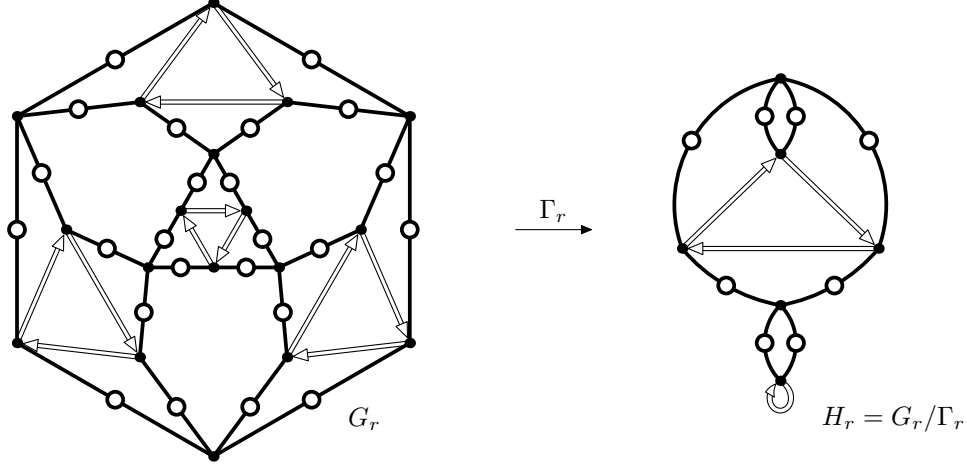


FIG. 4.3. A primitive graph G_r which is 3-connected. Let Γ_r be the semiregular subgroup of $\text{Aut}(G_r)$ generated by a 120° rotation. It defines the quotient $H_r = G_r/\Gamma_r$ which is not primitive (contains articulations and 2-cuts).

Proof. It follows from the fact that the graphs H_{s-1}, \dots, H_r are uniquely determined, since no half-edges are expanded till H_r . \square

LEMMA 4.2. *If H_s is expandable to H , then the core of H_s is expanded to some block or articulation of H .*

Proof. The graph H_s consists of the core together with some pendant edges, loops and half-edges. By Lemma 3.21, the core is preserved as an articulation/block in all graphs H_s, \dots, H_0 . The core can be only changed by replacing of its colored edges by edge-quotients. Since H_s is expandable to H_0 , the expanded core is isomorphic to some block or articulation of H . \square

In Step 4, we test all possible positions of the core in H . (We have $\mathcal{O}(n)$ possibilities, so we run the dynamic programming algorithm multiple times.) In what follows, we have the core fixed in H as well.

In Step 5, we apply on H reductions with lists, described in Section 4.3. In Step 6, we test whether some choices from these lists are compatible with the graph H_s .

4.2. Catalog of Atoms. During the reduction phase of the algorithm, we construct the following *catalog of atoms* forming a database of all discovered atoms. These atoms arise in three ways: atoms of G_0, \dots, G_{r-1} , atoms in half-quotients of these atoms, and atoms in the reductions of the quotients $H_r = G_r/\Gamma_r$. We are not very concerned with a specific implementation of the algorithm, so the purpose of this catalog is to simplify description.

For each isomorphism class of atoms represented by an atom A , we store the following information in the catalog:

- The atom A .

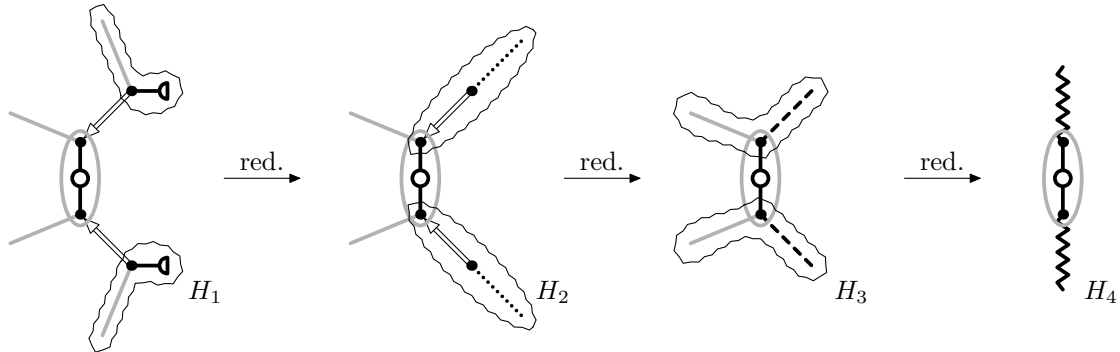


FIG. 4.4. The graph H_1 is one quotient of G_1 from Fig. 3.8. We further reduce it to H_3 with respect to the core block depicted in gray. Notice that H_1 and H_2 only contain block atoms.

- The corresponding colored edge of a given type representing the atom in the reduction.
- If A is an atom of G_0, \dots, G_{r-1} , the unique edge- and loop-quotients of A and information about its half-quotients.

For an overview of adding an atom A into the catalog, see Algorithm 1.

Storing Star Block Atoms. Let A be a star block atom. We store it in the catalog *partially expanded* which works as follows. By the definition, A consists of a vertex with attached edges, loops and half-edges. If some edge corresponds to a star block atom S , we replace it with the edges of S . Similarly, if some loop corresponds to the loop-quotient Q of a dipole D , we replace it by the loops of Q . We repeat this till all pendant edges of A correspond to block atoms and all loops of A correspond to loop-quotients of proper atoms. On the other hand, the half-edges of A may correspond to half-quotients of both proper atoms and dipoles.

Storing Dipoles. Let A be a dipole in G_0, \dots, G_{r-1} . By Lemma 3.20, it can have exponentially many non-isomorphic half-quotients. On the other hand, they are well described in Section 3.4, so we can generate all of them from the dipole when needed.

We store this dipole A in the catalog *partially expanded* which works as follows. Almost all edges of A correspond to proper atoms, while at most one edge corresponds to a dipole D . (At most one since from the definition, a dipole D with $\partial D = \{u, v\}$ consists of all edges between u and v .) If one edge corresponds to D , we replace it in A with the edges of the dipole D . And if one of these edges of D again correspond to some dipole D' , we proceed further with the expansion.

Notice that by the definition of the reduction, all colored edges of D have different colors than the edges

Algorithm 1 The subroutine for adding an atom into the catalog

Require: An atom A .

Ensure: If A is not contained in the catalog, then it is added. A colored halvable/undirected/directed edge corresponding to A is given.

```

1: if  $A$  is a star block atom then
2:   while  $A$  contains a pendant edge  $e'$  of a star block atom  $S$  do
3:     Replace  $e'$  with the edges of  $S$ .
4:   while  $A$  contains a loop  $e'$  of the loop-quotient of a dipole  $D$  do
5:     Replace  $e'$  with the loops of the loop-quotient of  $D$ .
6: if  $A$  is a dipole then
7:   while  $A$  contains an edge  $e'$  corresponding to a dipole  $D$  do
8:     Replace  $e'$  with the edges of  $D$ .
9: We test whether  $A$  is contained in the catalog using Lemma 4.4.
10: if  $A$  is contained in the catalog then
11:   return The corresponding colored edge representing  $A$ .

12: We determine the symmetry type of  $A$  using Lemmas 3.10 and 3.11.
13: We assign an edge  $e$  of a new color of the corresponding type to  $A$ .

14: if  $A$  is an atom of  $G_0, \dots, G_r$ . then
15:   We compute the edge-quotient of  $A$  and the loop-quotient of  $A$  (if  $A$  is not a block atom).
16:   if  $A$  is a dipole consisting of exactly two halvable edges of the same color then
17:     We add the half-quotient of  $A$  with one loop to the list of half-quotients.

18: if  $A$  is a halvable proper atom then
19:   We compute all half-quotients  $Q$  of  $A$  by Lemma 3.14.
20:   for each half-quotient  $Q$  do
21:     Apply the reduction series on  $Q$  with respect to the block containing  $\partial Q$ , constructing a primitive graph  $Q'$ .
22:     Add all detected atoms to the catalog and replace them by the corresponding colored edges.
23:     Add  $Q'$  to the catalog, as a half-quotient of  $A$ .

24: return The assigned colored edge  $e$  corresponding to  $A$ .

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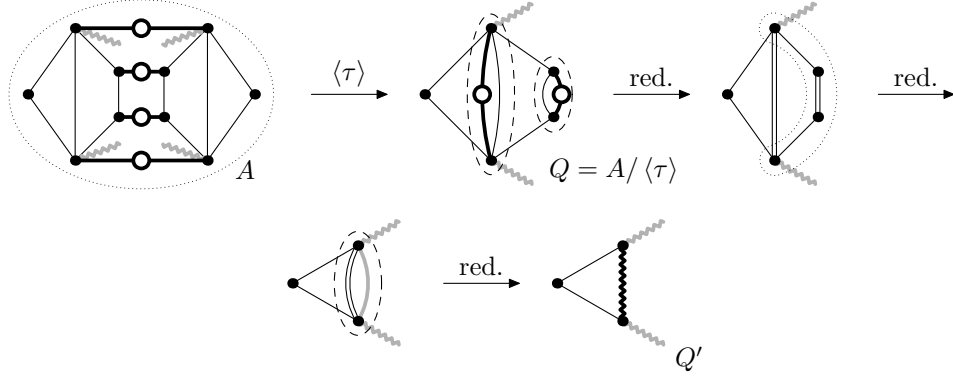


FIG. 4.5. A proper atom A with a half-quotient Q generated by 180° rotation τ . A reduction series is applied on Q which adds further atoms to the catalog and the primitive graph Q' .

of A . Therefore the half-quotients of the original dipole A are exactly the same as the half-quotients of the partially expanded dipole A . The reason for this expansion is that every half-quotient of the partially expanded dipole A consists of loops and half-edges attached to one vertex, where each loop and each half-edge is expanded into one block (with attached single pendant edges, half-edges and loops).

Further, if a halvable dipole A consists of exactly two edges of the same color, we compute its half-quotient consisting of just the single loop attached, and we add this quotient to the catalog. The reason is that this quotient behaves exactly as the loop-quotient of some proper atom.

Storing Proper Atoms. If A is not a dipole, we compute the list of all its pairwise non-isomorphic half-quotients, and store them in the catalog in the following way. A half-quotient Q of A might not be primitive. Therefore, we apply a reduction series on Q , and add all atoms discovered by the reduction to the catalog. (We do not compute their half-quotients. They are never realized unless these atoms are directly found in G as well.) When the reduction series finishes, this half-quotient is reduced to a primitive graph Q' . Naturally, the block containing ∂Q , being a single vertex of the half-quotient, behaves like the central block in the definition of atoms, i.e., it is never reduced. The reduced half-quotient Q' is either essentially 3-connected, a cycle with attached single pendant edges, or K_2 with a single pendant edge or half-edge attached. See Fig. 4.5 for an example.

Total Size of Catalog. Next, we prove that the catalog is not too large.

LEMMA 4.3. *Assuming (P2), the catalog contains polynomially many atoms and half-quotients.*

Proof. First we deal with the number of atoms in G_0, \dots, G_r . Notice that by replacing an interior of an atom, the total number of vertices and edges is decreased; the interiors of atoms in each G_i contain at least two vertices and edges in total and are pairwise disjoint (see Lemma 3.3). Thus we add a linear number of atoms of G_0, \dots, G_r to the catalog, of total linear size.

By (P2), there are polynomially many possible quotients H_r , in each we encounter linearly many atoms when reducing to H_s . So we add polynomially many atoms to the catalog.

By (P2), each proper atom A has polynomially many half-quotients, for different semiregular involutions of $\text{Aut}(A)$. So, we have in total polynomially many half-quotients, each containing at most linearly many atoms in its reduction series. And by Lemma 3.13 we have the unique edge- and loop-quotient. So again, the total number of atoms and quotients added to the catalog is polynomial. \square

Catalog Queries. Throughout the algorithm, we repeatedly ask *queries* whether some atom or some of its quotients is contained in the catalog, and if so, we retrieve the corresponding colored edge/loop/half-edge.

LEMMA 4.4. *Assuming (P3*), each catalog query can be answered in polynomial time.*

Proof. By Lemma 4.3, we need to test graph isomorphism for the input atom/quotient and polynomially many atoms/quotients in the catalog. If the input is an atom of an edge-quotient, we use Lemma 3.9. If it is a loop- or a half-quotient, then it is a primitive graph and we use Lemma 3.6. \square

4.3. Reductions with Lists. In this section, we describe Steps 5 and 6 of the diagram in Fig. 4.2. By Lemma 4.1, we need to test whether H_s is expandible to H . We approach this in the opposite way, by applying a reduction series on H with respect to the core defining $\mathcal{H}_0, \dots, \mathcal{H}_t$. As already discussed in

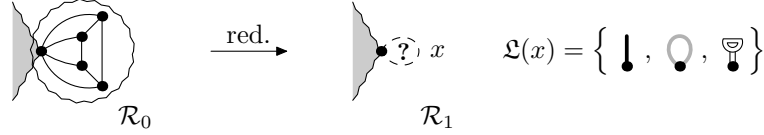


FIG. 4.6. Let x be the pendant element corresponding to the pendant block of H depicted in Fig. 4.1. Then $\mathfrak{L}(x)$ contains three different members if all three atoms depicted in Fig. 4.1 are contained in the catalog.

Section 4.1, we do not know which parts of G project to different parts of H . Therefore each \mathcal{H}_i is a set of graphs, and \mathcal{H}_i is a set of primitive graphs. We then determine expandability of H_s by testing whether $H_s \in \mathcal{H}_i$.

Since each set \mathcal{H}_i can contain a huge number of graphs, we represent it implicitly in the following manner. Each \mathcal{H}_i is represented by one graph \mathcal{R}_i with some colored edges and with so-called pendant elements attached to some vertices.

Pendant Elements with Lists. A pendant element x in \mathcal{R}_i corresponds to a block atom in \mathcal{R}_j for some $j < i$, which is reduced in \mathcal{R}_{j+1} . When pendant elements are fully expanded, they correspond to block part of H with pairwise disjoint interiors. We use the name pendant element since it may represent a pendant edge of some color, several loops of some other colors, and several half-edges of some other colors.

Each pendant element x is equipped with a *list* $\mathfrak{L}(x)$ whose *members* are possible realizations of the corresponding block atom by the quotients from the catalog. Each graph of \mathcal{H}_i is created for \mathcal{R}_i by replacing the pendant elements by some choices of edges, loops and half-edges from their lists. The list $\mathfrak{L}(x)$ of a pendant element x contains an edge/loop/half-edge if and only if it is possible to expand this edge/loop/half-edge to the graph isomorphic to the block part corresponding to x . For an example, see Fig. 4.6. According to Lemma 4.3, we have polynomially many atoms, and so the size of each list is polynomial in size.

LEMMA 4.5. *Each list $\mathfrak{L}(x)$ contains at most one edge. Further, if two lists share an edge or a loop, their pendant elements correspond to isomorphic block parts in H .*

Proof. Two atoms have the isomorphic edge-quotients if and only if they are isomorphic. Therefore each list $\mathfrak{L}(x)$ contains at most one edge.

If a pendant element x is fully expanded, it corresponds to one block part of H . Suppose that an edge- or a loop-quotient belongs to $\mathfrak{L}(x)$. If it is fully expanded, then it has to be isomorphic to this block part. But according to Lemma 3.13, the expansions of edge- and loop-quotients are deterministic since half-edges are never encountered. Therefore the corresponding block part in H is uniquely determined. \square

One list may contain several loops, for which identifying of the vertices of the boundaries constructs

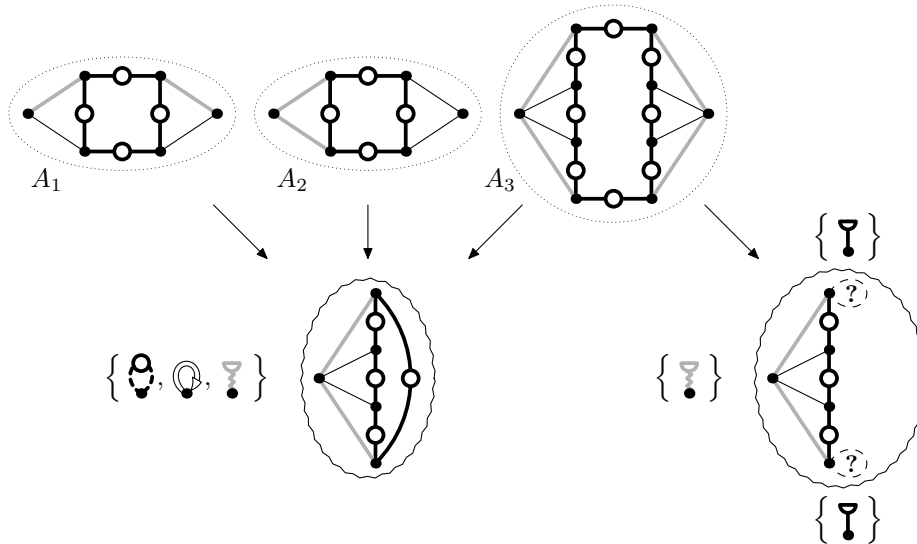


FIG. 4.7. Two block atoms corresponding to pendant elements with depicted lists. On the left, the list has the loops corresponding to A_1 and A_2 and the half-edge corresponding to A_3 . On the right, the list only contains the half-edge of A_3 .

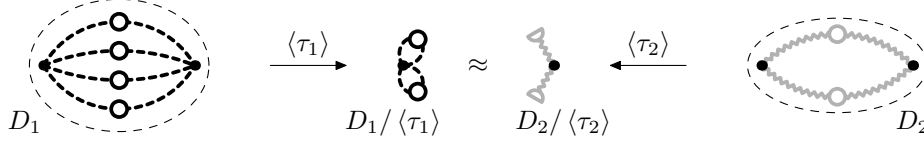


FIG. 4.8. An example of two dipoles D_1 and D_2 having isomorphic half-quotients. Consider the atoms A_1 and A_3 from Fig. 4.7, for which the loop-quotient of A_1 is isomorphic to a half-quotient of A_3 . Let D_1 consist of four edges corresponding to A_1 and let D_2 consist of two edges corresponding to A_3 . Then the half-quotient $D_1 / \langle \tau_1 \rangle$ can be expanded to a graph isomorphic to an expansion of the half-quotient $D_2 / \langle \tau_2 \rangle$.

identical graphs; see Fig. 4.7. Similarly, a list may contain several half-edges; see Fig. 4.8. Because of the second part of Lemma 4.5, the loops pose no problem. On the other hand, one half-edge may be contained in lists of several different pendant elements which are expanded to non-isomorphic subgraphs in H ; see Fig. 4.7. This creates the main difficulty for our algorithm, leading to the bottleneck in form of a slow subroutine requiring time $\mathcal{O}^*(2^{e(H)/2})$.

Reductions with Lists. We want to compute the reduction series with lists $H = \mathcal{R}_0, \mathcal{R}_1, \dots, \mathcal{R}_t$ ending with a primitive graph \mathcal{R}_t with attached pendant elements with computed lists. We construct \mathcal{R}_0 by replacing all pendant edges and loops by pendant elements with singleton lists.

Suppose that we know \mathcal{R}_i , and we want to apply one step of the reduction and compute \mathcal{R}_{i+1} . We find all atoms in \mathcal{R}_i . We define atoms with respect to the chosen core in H , and we work with pendant elements as with pendant edges. Further, we consider only star block atoms consisting only of an articulation with all its descendants attached in form of reduced pendant elements. This means that we postpone reduction of star block atoms till all their descendants are reduced first. (The same modification could be applied in all reductions as well, but it is important here.)

To construct \mathcal{R}_{i+1} from \mathcal{R}_i , we proceed with the following:

- We replace dipoles and proper atoms by edges of the corresponding colors from the catalog. A proper atom might have pendant elements attached to its interior, but these pendant elements are always realized by edges corresponding to the edge-quotients of some block atoms. Therefore, we can replace the pendant elements with lists by the unique edges from these lists, and if some list contains no edge, we stop the reduction. We run a catalog query and if the dipole or proper atom is not contained in the catalog, we halt the reduction procedure.
- We replace block atoms by pendant elements with constructed lists. If some list is empty, we again halt the reduction.

It remains to describe the construction of the lists for the created pendant elements.

Computing Lists. Let A be a block atom in \mathcal{R}_i , replaced by a pendant element x in \mathcal{R}_{i+1} , and we want to compute $\mathfrak{L}(x)$. We compute $\mathfrak{L}(x)$ from the lists of the pendant elements attached to A . Suppose that A has pendant elements y_1, \dots, y_p attached. For each member of $\mathfrak{L}(x)$, we remember which members of $\mathfrak{L}(y_1), \dots, \mathfrak{L}(y_p)$ have to be chosen for its expansion.

LEMMA 4.6. *Let A be a non-star block atom in \mathcal{R}_i . Assuming (P3), we can compute the list $\mathfrak{L}(x)$ of the pendant element x corresponding to A in polynomial time.*

Proof. We iterate over quotients in the catalog which are K_2 with a single pendant edge, essentially cycles, or essentially 3-connected graphs by Lemma 3.7. Let Q be such a quotient. For $u \in \partial A$, we put $\mathfrak{L}(u) = \partial Q$. For each single pendant element y of A attached at u , we construct $\mathfrak{L}(u)$ consisting of all vertices $v \in V(Q)$ such that the pendent edge/loop/half-edge attached at v belongs to $\mathfrak{L}(y)$. We remove all pendant elements attached at A and all pendant edges/loops/half-edges attached at Q .

By definition of pendant elements, it is possible to expand Q to the block part corresponding to A if and only if there exists a list-compatible isomorphism $A \xrightarrow{\mathfrak{L}} Q$. If Q is K_2 or a cycle, we test it trivially. If Q is 3-connected, we test it using (P3). If $A \xrightarrow{\mathfrak{L}} Q$, we add the pendant edge/loop/half-edge representing this quotient to the list $\mathfrak{L}(x)$, and we remember the constructed isomorphism $A \xrightarrow{\mathfrak{L}} Q$. See Fig. 4.9 for an example. \square

On the other hand, if A is a star block atom, we compute its list by a slow subroutine. If this slow subroutine can be avoided and the list for A can be computed in polynomial time, the entire meta-algorithm of Theorem 1.3 runs in polynomial time.

LEMMA 4.7. *Let A be a star block atom in \mathcal{R}_i . We can compute the list $\mathfrak{L}(x)$ of the pendant element x*

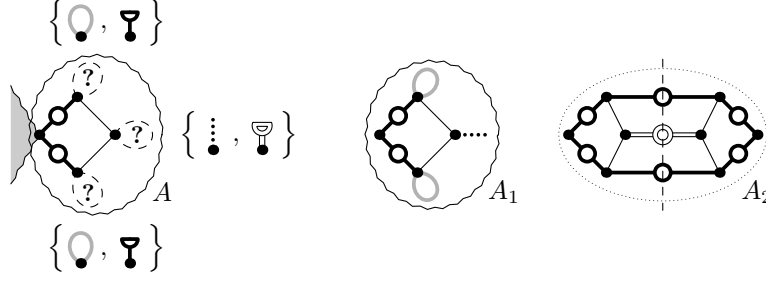


FIG. 4.9. On the left, a non-star block atom A in \mathcal{R}_i with depicted lists of its pendant elements. On the right, two possible atoms from the catalog having a quotient for which there exists a list-compatible isomorphism from A . So the list of the pendant element replacing A in \mathcal{R}_{i+1} contains the pendant edge corresponding to the edge-quotient of the block atom A_1 and the half-edge corresponding to a half-quotient of the proper atom A_2 .

corresponding to A in time $\mathcal{O}^*(2^{e(H)/2})$.

Proof. Each star block atom of \mathcal{R}_i corresponds either to the edge-quotient of a star block atom, or to the loop- or a half-quotient of a dipole. Lemma 3.20 states that a dipole can have exponentially many pairwise non-isomorphic half-quotients, we iterate over all of them which gives $2^{e(H)/2}$ part in the complexity bound. Since we postpone reduction of star block atoms, all pendant elements of A necessarily correspond to non-star block atoms in some \mathcal{R}_j , for $j < i$, so each pendant element corresponds in H to one subtree of blocks attached at the vertex of A .

Case 1: Dipoles. We iterate over all partially expanded dipoles in the catalog and try to add them to the list $\mathfrak{L}(x)$. Let D be a partially expanded dipole, recall that all edges of D correspond to proper atoms.

We test whether the lists of the pendant elements attached to the star block atom A are compatible with the loop-quotient of D . Each loop of this loop-quotient corresponds to the loop-quotient of some proper atom which is either a cycle with attached single pendant edges, or essentially 3-connected by Lemma 3.8. Therefore, it corresponds to exactly one pendant element in A . By Lemma 4.5, each loop belongs only to lists of pendant elements of type. Therefore, we just need to compare the number of loops in each color class with the number of lists containing this colored loop. If these numbers match, we add the loop representing the loop-quotient of D to $\mathfrak{L}(x)$.

Then we iterate over all half-quotients of D . By Lemma 3.20, let Q be one of its at most $2^{e(H)/2}$ possible quotients. Recall from Section 3 that an edge of D projects either to a half-edge, or together with another edge of D of the same color and type to one loop. So each Q consists of loops and half-edges attached to a vertex. Since all edges of D correspond to proper atoms, each loop and each half-edge has to be matched to one pendant element of A .

Therefore, we test existence of a perfect matching in the following bipartite graph: One part is formed by the loops and the half-edges of Q , and the other part is formed by the pendant elements of A . A loop/half-edge is adjacent to a pendant element, if and only if the corresponding list contains this loop/half-edge. Each perfect matching defines one assignment of the loops and half-edges of Q to the pendant elements of A . See

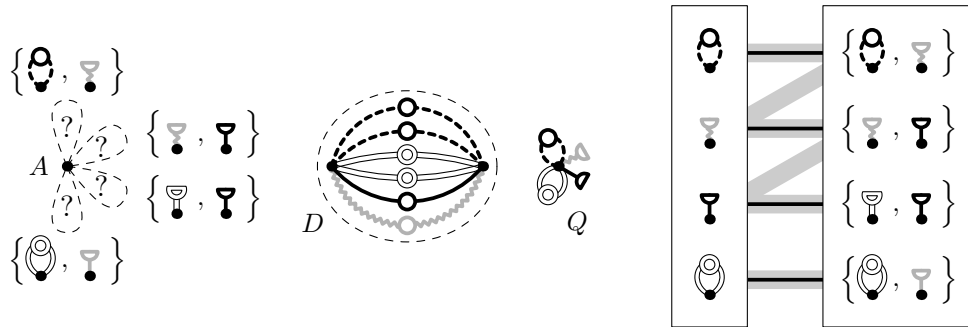


FIG. 4.10. The half-edge corresponding to a half-quotient of the dipole D belongs to the list $\mathfrak{L}(x)$ of a pendant element x replacing A because there exists a perfect matching between the loops and half-edges of the half-quotient Q of D and the lists of pendant elements of A .

Fig. 4.10 for an example. We add the half-edge corresponding to a half-quotient of D to $\mathfrak{L}(x)$ if and only if there exists a perfect matching for at least one half-quotient Q of D .

Case 2: Star Block Atoms. We iterate over all partially expanded star block atoms of the catalog, let S be one of them. The star block atom S consists of one vertex with attached pendant edges (corresponding to non-star block atoms), loops (corresponding to the loop-quotients of proper atoms) and half-edges. Some of these half-edges correspond to dipoles, and some to proper atoms. Let h_1, \dots, h_d be the half-edges corresponding to partially expanded dipoles D_1, \dots, D_d from the catalog. We construct all expanded edge-quotients Q of S by replacing h_1, \dots, h_d by all possible choices of half-quotients Q_1, \dots, Q_d of D_1, \dots, D_d . In total, we have at most $2^{e(H)/2}$ different expanded edge-quotients Q of S .

All pendant edges of Q correspond to non-star block atoms, and all loops and half-edges correspond to loop- and half-quotients of proper atoms. Therefore, every edge, loop and half-edge attached in Q has to be matched to one pendant element of A . Similarly as above, for each expanded edge-quotient Q , we test whether there exists a perfect matching between edges, loops and half-edges of Q and the lists of pendant elements of A . We add the edge representing the edge-quotient of the star block atom S to $\mathfrak{L}(x)$, if and only if there exists a perfect matching for some expanded edge-quotient Q of S .

The procedure computes the list $\mathfrak{L}(x)$ correctly since we test all possible quotients from the catalog, and for each quotient we test all possibilities how it could be matched to A . For each quotient Q , the running time is clearly polynomial, and we have $\mathcal{O}^*(2^{e(H)/2})$ quotients. \square

Algorithm 2 gives the pseudocode for computation of the list $\mathfrak{L}(x)$ of a pendant element x replacing an atom A . If the returned list is empty, we halt the reduction; either H_s is not expandable to H , or we have chosen a wrong core in H .

Testing Expandability. The reduction with lists ends with a primitive graph \mathcal{R}_t with lists. For one particular choice of a core, \mathcal{R}_t represents the set of graphs \mathcal{H}_t to which H can be reduced.

In the following, we denote by $\mathcal{R}_t \xrightarrow{\mathfrak{L}} H_s$ existence of a *list-compatible isomorphism* which preserves colors and orientations of edges and maps pendant elements x of \mathcal{R}_t into pendant edges, loops and half-edges of H_s such that $\pi(x) \in \mathfrak{L}(x)$. (It corresponds to a list-compatible isomorphism defined in Section 1 when pendant elements/edges/loops/half-edges are removed, as described in the proof of Lemma 4.6.)

LEMMA 4.8. *The graph H_s is expandable to H_0 which is isomorphic to H if and only if $\mathcal{R}_t \xrightarrow{\mathfrak{L}} H_s$ for some choice of the core in H .*

Proof. Suppose that $\mathcal{R}_t \xrightarrow{\mathfrak{L}} H_s$ for some choice of the core. By the definition, every pendant element x of \mathcal{R}_t can be replaced by any member of $\mathfrak{L}(x)$ which can be fully expanded to the block part in H corresponding to x . The list-compatible isomorphism chooses for pendant elements of \mathcal{R}_t realization by edges, loops and half-edges which is compatible with the computed quotient of H_s .

In more detail, we first expand edges in H_s, \dots, H_{r+1} by the unique edge-quotients to reach H_r , this has to be compatible with the sequence of replacements defined by $\mathcal{R}_t \xrightarrow{\mathfrak{L}} H_s$. Then we do replacements in the manner of Theorem 3.18, and construct the expansions H_{r-1}, \dots, H_0 . Since we expand according to the list-compatible isomorphism $\mathcal{R}_t \xrightarrow{\mathfrak{L}} H_s$, the constructed graph H_0 is isomorphic to H .

On the other hand, suppose that H_s is expandable to H_0 which is isomorphic to H . Then according to Lemma 3.21, the core of H_s is preserved in H , so it has to correspond to some block or to some articulation of H , which we choose as the core of H . Since there exists a sequence of replacements from H_s which constructs H_0 , this sequence of replacements is possible in \mathcal{R}_t . Thus $\mathcal{R}_t \xrightarrow{\mathfrak{L}} H_s$. \square

LEMMA 4.9. *Assuming (P3), we can test whether H_s is expandable to H_0 which is isomorphic to H in time $\mathcal{O}^*(2^{e(H)/2})$.*

Proof. We iterate over all choices of the core in H . For each, we compute the reduction series with lists $H = \mathcal{R}_0, \dots, \mathcal{R}_t$, using Algorithm 2 and Lemmas 4.6 and 4.7. We modify both graphs \mathcal{R}_t and H_s similarly as in the proof of Lemma 4.6. For each pendant element x of \mathcal{R}_t attached at u , we put $\mathfrak{L}(u)$ be the set of all vertices of $V(H_s)$ having an attached pendant edge/loop/half-edge which belongs to $\mathfrak{L}(x)$, and we remove x . We remove all pendant edges/loops/half-edges of H_s .

Then we test whether $\mathcal{R}_t \xrightarrow{\mathfrak{L}} H_s$. It is trivial to deal with the cases when H_s or \mathcal{R}_t are cycles, K_2 or K_1 . Otherwise by Lemma 3.5, both H_s and \mathcal{R}_t are 3-connected graphs, and we test $\mathcal{R}_t \xrightarrow{\mathfrak{L}} H_s$ using (P3). By Lemma 4.8, this subroutine is correct and runs in time $\mathcal{O}^*(2^{e(H)/2})$. \square

4.4. Proof of The Main Theorem. Now, we are ready to establish the main algorithmic result of the paper; see Algorithm 3 for the pseudocode. Assuming that a class \mathcal{C} satisfies (P1) to (P3), we show that REGULARCOVER can be solved for \mathcal{C} -inputs G in time $\mathcal{O}^*(2^{e(H)/2})$:

Algorithm 2 The subroutine for computing lists of pendant elements

Require: A block atom A of \mathcal{R}_i .

Ensure: The list $\mathfrak{L}(x)$ of the pendant element x replacing A in \mathcal{R}_{i+1} .

- 1: Initiate the empty list $\mathfrak{L}(x)$.
 - 2: **if** A is a non-star block atom **then**
 - 3: Iterate over all quotients from the catalog.
 - 4: **for** each quotient Q from the catalog **do**
 - 5: For each pendant element x of A attached at u , set $\mathfrak{L}(u)$ to all vertices $V(Q)$ having an attached pendant edge/loop/half-edges belonging to $\mathfrak{L}(x)$, and remove x .
 - 6: Remove all pendant edges/loops/half-edges in Q , and test $A \xrightarrow{\mathfrak{L}} Q$ (trivially or using (P3)).
 - 7: If some list-compatible isomorphism exists, we add the edge/loop/half-edge of Q to $\mathfrak{L}(x)$ together with this list-compatible isomorphism.
 - 8: **if** A is a star block atom **then**
 - 9: Iterate over all partially expanded dipoles D and star block atoms S in the catalog.
 - 10: **for** each partially expanded dipole D **do**
 - 11: Test whether the loop-quotient of D matches the lists; if yes, then add the loop representing D to $\mathfrak{L}(x)$.
 - 12: Iterate over all half-quotients Q of D .
 - 13: **for** each half-quotient Q **do**
 - 14: Test existence of a perfect matching between the loops and half-edges of Q and the lists of the pendant elements of A .
 - 15: If a perfect matching exists, add the half-edge of D to $\mathfrak{L}(x)$ together with this half-quotient Q and this matching, and proceed with the next dipole.
 - 16: **for** each partially expanded star block atom S **do**
 - 17: Compute all expanded edge-quotients Q of S by replacing the half-edges h_1, \dots, h_d corresponding to the dipoles by all possible combinations of their half-quotients Q_1, \dots, Q_d .
 - 18: **for** each expanded edge-quotient Q **do**
 - 19: Test existence of a perfect matching between the edges, loops and half-edges of Q and the lists of the pendant elements of A .
 - 20: If a perfect matching exists, then add the edge of S to $\mathfrak{L}(x)$ with this expanded edge-quotient Q and this matching, and proceed with the next star block atom.
 - 21: **return** The constructed list $\mathfrak{L}(x)$.
-

Proof. [Theorem 1.3] We recall the main steps of the algorithm and discuss their time complexity. The reduction series G_0, \dots, G_r can be computed in polynomial time, by Lemmas 3.15 and 4.4. We reach in G_r one of primitive graphs characterized in Lemma 3.5. If G_r is essentially 3-connected, the property (P2) ensures that there are polynomially many semiregular subgroups Γ_r of $\text{Aut}(G_r)$ which can be computed in polynomial time. If G_r is K_2 with attached single pendant edges or essentially a cycle, it is true as well.

For each of these subgroups Γ_r , we compute the quotient $H_r = G_r/\Gamma_r$. Then we compute the reduction series H_r, \dots, H_s , again in polynomial time using Lemma 4.4. Using Lemma 4.9, we test in time $\mathcal{O}^*(2^{e(H)/2})$ whether H_s is expandable to H_0 which is isomorphic to H . We output “yes” if and only if $H_r = G_r/\Gamma_r$ is expandable to H_0 isomorphic to H for at least one the subgroups Γ_r .

To certify the “yes” outputs, we construct the semiregular subgroup $\Gamma \leq \text{Aut}(G)$ such that $G/\Gamma \cong H$ as follows. If $\mathcal{R}_t \xrightarrow{\mathfrak{L}} H_s$ for some choice of the core in H , this list-compatible isomorphism describes how to expand H_s to H_0 which is isomorphic to H using Theorem 3.18. This expansion replaces edges, loops and half-edges with edge-quotients, loop-quotients and some choices of half-quotients. The expansion towards H_r is deterministic since no half-edges are replaced.

We expand H_r, \dots, H_0 , together with constructing group extensions $\Gamma_{r-1}, \dots, \Gamma_0$ of Γ_r , where Γ_i is a semiregular subgroup of $\text{Aut}(G_i)$. When G_{i+1} is expanded to G_i , we replace some edges with interiors of some atoms. In the common parts, we define the actions of Γ_i and Γ_{i+1} the same. It remains to define the action of Γ_i on the interiors of these atoms in such a way that $G_i/\Gamma_i = H_i$. When some orbit of atoms is projected to the edge- or loop-quotients, then we isomorphically swap their interiors in Γ_i the same as

Algorithm 3 The meta-algorithm for regular covers – REGULARCOVER**Require:** A graph G of \mathcal{C} satisfying (P1), (P2) and (P3), and a graph H .**Ensure:** A semiregular subgroup $\Gamma \leq \text{Aut}(G)$ such that $G/\Gamma \cong H$ if it exists.

- 1: Compute the reduction series G_0, \dots, G_r ending with the primitive graph G_r .
- 2: During the reductions, we use Algorithm 1 to add atoms and their quotients into the catalog, and to replace them with colored edges.
- 3: Using (P2), we compute all semiregular subgroups Γ_r of $\text{Aut}(G_r)$.
- 4: **for** each semiregular subgroup Γ_r **do**
- 5: Compute the quotient $H_r = G_r/\Gamma_r$.
- 6: Choose, say, the central block/articulation of H_r as the core.
- 7: Compute the reduction series H_r, \dots, H_s with respect to the core.
- 8: During the reductions, we use Algorithm 1 to add atoms into the catalog, and to replace them with colored edges.
- 9: **for** each guessed position of the core in H **do**
- 10: Compute the reduction series with lists $H = \mathcal{R}_0, \dots, \mathcal{R}_t$.
- 11: **to** compute \mathcal{R}_{i+1} from \mathcal{R}_i **do**
- 12: **for** each proper atom or dipole A in \mathcal{R}_i **do**
- 13: **if** A is a proper atom **then**
- 14: Replace its pendant elements with the unique pendant edges from their lists. If some list contains no pendant edge, halt and test for other choices of the core in H .
- 15: Replace A with a colored edge using the catalog. Halt and test for other choices of the core in H if A is not in the catalog.
- 16: **for** each block atom A in \mathcal{R}_i **do**
- 17: Replace it by a pendant element x whose list $\mathfrak{L}(x)$ is computed using Algorithm 2. Halt and test for other choices of the core in H if $\mathfrak{L}(x)$ is empty.
- 18: Test $\mathcal{R}_t \xrightarrow{\mathfrak{L}} H_s$ using Lemma 4.9.
- 19: **if** $\mathcal{R}_t \xrightarrow{\mathfrak{L}} H_s$ **then**
- 20: Using the lists, compute the expansions H_{s-1}, \dots, H_0 such that $H_0 \cong H$.
- 21: Using Theorem 3.18, compute the group extensions $\Gamma_{r-1}, \dots, \Gamma_0 = \Gamma$ to the interiors of expanded edges, loops and half-edges. For half-edges, use involutions τ on interiors of replacing half-quotients.
- 22: By Lemma 4.8, $G/\Gamma \cong H$, so the group Γ defines the regular covering projection $p : G \rightarrow H$.
- 23: **return** The semiregular subgroup $\Gamma \leq \text{Aut}(G)$.
- 24: **return** The graph G does not regularly cover the graph H .

the corresponding edges are swapped in Γ_{i+1} . For an orbit which is projected to half-quotients, we further compose some automorphisms of Γ_i with the half-quotient defining semiregular involution τ on their interior (when the corresponding edges are flipped in Γ_i). For more details, see [26], proofs of Lemma 4.7 and Theorem 1.3 therein.

It remains to argue correctness of the algorithm. First suppose that the algorithm succeeds. We construct a semiregular subgroup Γ of $\text{Aut}(G)$. By Lemma 4.8, some H_s is expandible to H_0 which is isomorphic to H . By Theorem 3.18, we get that $G/\Gamma \cong H$ which proves that G regularly covers H . On the other hand, suppose that there exists a semiregular Γ such that $H \cong G/\Gamma$. Then Γ corresponds to the unique semiregular subgroup Γ_r on G_r which is one of the semiregular subgroups tested by the algorithm. Therefore H_r has to be expandible to H_0 isomorphic to H , and we detect this correctly according to Lemma 4.8. \square

Next, we prove two corollaries. The first corollary states that if G is 3-connected, H is 2-connected or $k = |G|/|H|$ is odd, the meta-algorithm can avoid the slow subroutine of Lemma 4.7 and can be modified to run in polynomial time.

Proof. [Corollary 1.4] If G is 3-connected, then it is primitive, so $G = G_r$. Therefore, we compute all quotients $H_r = G_r/\Gamma_r$, and test using Lemma 3.16 whether $H_r \cong H$. No reduction with lists needs to be applied. If H is 2-connected, no pendant elements are created the reduction of H with lists, so the slow

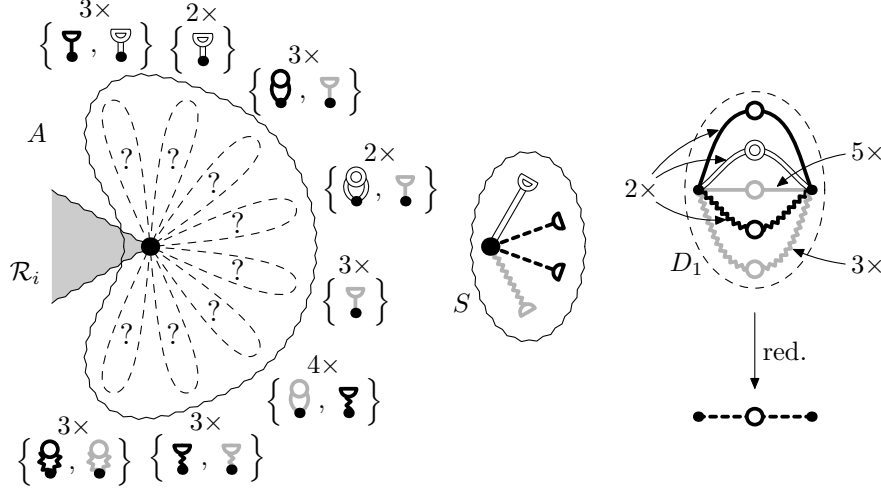


FIG. 5.1. On the left, a star block atom A in \mathcal{R}_i with 23 attached pendant elements, together with their lists and multiplicities. On the right, a star block atom S from the catalog which belongs to $\mathfrak{L}(x)$. The bold dashed edges correspond to the partially expanded dipole D_1 whose edges are depicted with multiplicities, the remaining colored edges correspond to proper atoms.

subroutine can be avoided and even the assumptions (P1), (P2) and (P3*) are sufficient.

If $|\Gamma| = |\Gamma_r|$ is odd, then no half-edges occur in H_r , and so according to Corollary 3.19, the expansion gives the unique graph H_0 . We just test whether $H_0 \cong H$. \square

Next, we prove that we can modify the meta-algorithm to output all regular quotients of G , with a polynomial-time delay.

Proof. [Corollary 1.5] We compute the reduction series $G = G_0, \dots, G_r$ and all semiregular subgroups Γ_r of $\text{Aut}(G_r)$. Next, we run all possible expansions of $H_r = G_r/\Gamma_r$ to H_0 using Theorem 3.18, by all possible choices of half-quotients. All half-quotients of proper atoms can be computed in polynomial time. For dipoles, we can easily generate them with polynomial-time delays. We output all constructed graphs H_0 . \square

5. Star Blocks Atoms with Lists. The bottleneck in the running time of the meta-algorithm of Theorem 1.3 is the single slow subroutine in Lemma 4.7, computing lists of pendant elements replacing star block atoms in \mathcal{R}_i . In this section, we give insights into this problem, which might lead to a faster algorithm for REGULARCOVER of planar graphs.

We show a combinatorial reformulation to finding a certain generalization of a perfect matching which we call IV-MATCHING. Here we describe a complete derivation of this problem, and in Conclusions we just give its combinatorial statement.

Instance. Figure 5.1 shows an example. Suppose that \mathcal{R}_i contains a star block atom A with attached pendant elements, each with a previously computed list and corresponding to a non-star block atom. We want to determine the list $\mathfrak{L}(x)$ of the pendant element x replacing A in \mathcal{R}_{i+1} . The following are the candidates for members of $\mathfrak{L}(x)$:

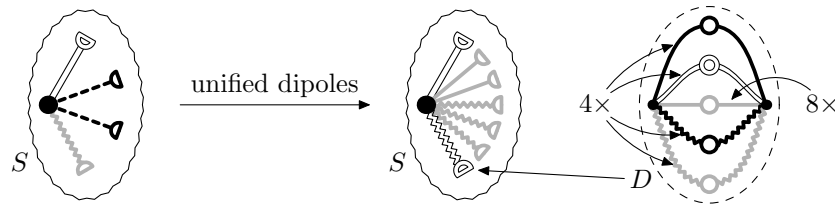


FIG. 5.2. We apply the unification on the example in Fig. 5.1 as follows. We unify both occurrences of the dipole D_1 into the dipole D . Since the colored classes of gray edges have odd sizes, we attach one half-edge per dipole from each class directly to S .

- Each loop corresponding to the loop-quotient of a dipole D .
- Each half-edge corresponding to a half-quotient of a dipole D .
- Each edge corresponding to the edge-quotient of a star block atom S .

Since loop-quotients of dipoles are uniquely determined, we can easily test them and we can ignore them. The case of a half-quotient of a dipole D can be reduced to a star block atom S with a single half-edge attached corresponding to a half-quotient of D . If S can be matched to A , we instead add the half-edge corresponding to a half-quotient of D to $\mathfrak{L}(x)$. Therefore, in the remainder of this section, we only deal with the case of a star block atoms S . We want to decide whether the edge corresponding to the edge-quotient of S belongs to $\mathfrak{L}(x)$. We shall assume that at least one half-edge attached in S corresponds to a dipole, otherwise the problem is trivial.

Outline. In Section 5.1, we simplify both A and S . In Section 5.2, we further apply size constraints to simplify them. In Section 5.3, we derive the IV-MATCHING problem.

5.1. Preprocessing Star Block Atoms. The star block atom S has several pendant edges, loops and half-edges attached. Further, we may assume that S is partially expanded (see Section 4.2), so its pendant edges correspond to non-star block atoms and its loops correspond to proper atoms. On the other hand, a half-edge can be of two types: either it corresponds to a half-quotient of a proper atom, or of a dipole. For example, in Fig. 5.1 we have two half-edges corresponding to proper atoms, and two half-edges corresponding to dipoles. Further, we may assume that all these dipoles are partially expanded (see Section 4.2), so all their edges correspond to proper atoms.

Unifying Dipoles. Recall that every half-quotient of a dipole consists of half-edges and loops attached to one vertex. Since S may contain multiple half-edges corresponding to half-quotients of dipoles, we want to unify them into one dipole D containing all their edges. (This may happen in the quotients; see Fig. 5.3.)

The issue is that this unification might introduce additional quotients of D as in Fig. 5.3. If two dipoles both contain an odd number of edges of one color, the unified dipole D has a half-quotient consisting of only loops of this color which is not possible in the case of half-quotients of two separated dipoles. There is an easy fix: we check each dipole and we remove one edge from each color class of odd size (of necessarily halvable edges) and attach the half-edge of this color directly to S . At least one half-edge of this color appears in every half-quotient of this dipole, so the possible half-quotients are not changed. In Fig. 5.2, we illustrate this preprocessing for the example in Fig. 5.1.

Non-halvable Edges of The Dipole. If the dipole D contains some non-halvable edges, then they are paired in every half-quotient of D and form loops. We remove them from D and attach the corresponding number of loops directly in S . After this step, the dipole D contains only even number of halvable edges in each color class.

Attached Pendant Edges and Loops. The star block atom S may have some pendant edges (corresponding to non-star block atoms) and loops (corresponding to proper atoms) attached. Therefore, each attached pendant edge/loop corresponds to exactly one pendant element of A . By Lemma 4.5, each is contained in list of only one type of pendant elements, all corresponding to isomorphic block parts in H . Therefore, we can arbitrarily assign pendant elements, remove them from A and remove these pendant edges and loops from S .

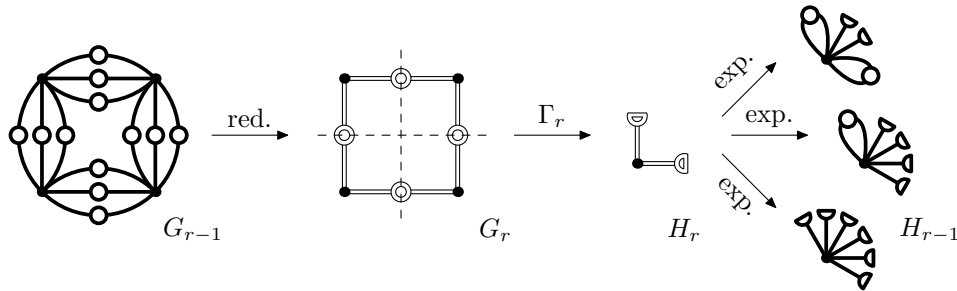


FIG. 5.3. For Γ_r generated by two reflections, the quotient H_r consists of a star block atom with two half-edges corresponding to dipoles. All three expansions H_{r-1} up to isomorphism are depicted. But it is not possible to expand H_r to the quotient with three attached loops.

Summary. By the preprocessing of S and A described above, we may assume the following. The star block atom S has only half-edges attached, all but one corresponding to proper atoms. The remaining half-edge corresponds to the unified dipole D having only color classes of even sizes of halvable edges corresponding to proper atoms.

For each pendant element x of A , the list $\mathfrak{L}(x)$ contains only half-edges (attached in S or in a half-quotient of D) and loops (corresponding to halvable edges of D).

5.2. Sizes and Chains. To simplify the problem further, we study sizes of atoms and their quotients. Let A be an atom and let Q be a quotient of this atom. Depending on the type of Q , we get:

- Q is the edge-quotient: Then $\mathbf{v}(Q) = \mathbf{v}(A)$ and $\mathbf{e}(Q) = \mathbf{e}(A)$.
- Q is the loop-quotient: Then $\mathbf{v}(Q) = \mathbf{v}(A) - 1$ and $\mathbf{e}(Q) = \mathbf{e}(A)$.
- Q is a half-quotient: Then $\mathbf{v}(Q) = \mathbf{v}(A)/2$ and $\mathbf{e}(Q) = \mathbf{e}(A)/2$.

Sizes of Expanded Subgraphs and Quotients. Throughout each reduction, we calculate how many vertices and edges are in all the atoms replaced by colored edges which we denote by $\hat{\mathbf{v}}$ and $\hat{\mathbf{e}}$. Initially, we put $\hat{\mathbf{v}}(e) = 0$ and $\hat{\mathbf{e}}(e) = 1$ for every edge $e \in \mathbf{E}(G_0)$. For a subgraph X , we define

$$\hat{\mathbf{v}}(X) := \mathbf{v}(X) + \sum_{e \in \mathbf{E}(X)} \hat{\mathbf{v}}(e), \quad \text{and} \quad \hat{\mathbf{e}}(X) := \sum_{e \in \mathbf{E}(X)} \hat{\mathbf{e}}(e).$$

When an atom A is replaced by an edge e in the reduction, we put $\hat{\mathbf{v}}(e) = \hat{\mathbf{v}}(\mathring{A})$ and $\hat{\mathbf{e}}(e) = \hat{\mathbf{e}}(\mathring{A})$. For a subgraph X of G_i , the numbers $\hat{\mathbf{v}}(X)$ and $\hat{\mathbf{e}}(X)$ are the numbers of vertices and edges when X is fully expanded.

We similarly define $\hat{\mathbf{v}}$ and $\hat{\mathbf{e}}$ for quotients and their subgraphs; the difference is that the quotients may contain half-edges. For a half-edge $h \in \mathbf{H}(X)$, created by halving an edge e , we put $\hat{\mathbf{v}}(h) = \hat{\mathbf{v}}(e)/2$ and $\hat{\mathbf{e}}(h) = \hat{\mathbf{e}}(e)/2$. For a subgraph X , we define

$$\hat{\mathbf{v}}(X) := \mathbf{v}(X) + \sum_{e \in \mathbf{E}(X)} \hat{\mathbf{v}}(e) + \sum_{h \in \mathbf{H}(X)} \hat{\mathbf{v}}(h), \quad \text{and} \quad \hat{\mathbf{e}}(X) := \sum_{e \in \mathbf{E}(X)} \hat{\mathbf{e}}(e) + \sum_{h \in \mathbf{H}(X)} \hat{\mathbf{e}}(h).$$

Sizes of Pendant Elements. We also inductively define $\hat{\mathbf{v}}$ and $\hat{\mathbf{e}}$ for pendant elements x and subgraphs X of $\mathcal{R}_0, \dots, \mathcal{R}_t$. Initially, we put $\hat{\mathbf{v}}(e) = 0$ and $\hat{\mathbf{e}}(e) = 1$ for every edge $e \in \mathbf{E}(\mathcal{R}_0)$. For a subgraph X of \mathcal{R}_i , let $\mathbf{P}(X)$ be the set of all pendant elements in X . We define

$$\hat{\mathbf{v}}(X) := \mathbf{v}(X) + \sum_{e \in \mathbf{E}(X)} \hat{\mathbf{v}}(e) + \sum_{y \in \mathbf{P}(X)} \hat{\mathbf{v}}(\mathring{y}), \quad \text{and} \quad \hat{\mathbf{e}}(X) := \sum_{e \in \mathbf{E}(X)} \hat{\mathbf{e}}(e) + \sum_{y \in \mathbf{P}(X)} \hat{\mathbf{e}}(y),$$

while for a pendant element x corresponding to an atom A in \mathcal{R}_i , we define $\hat{\mathbf{v}}(x) = \hat{\mathbf{v}}(A)$, $\hat{\mathbf{v}}(\mathring{x}) = \hat{\mathbf{v}}(\mathring{A}) = \hat{\mathbf{v}}(x) - 1$, and $\hat{\mathbf{e}}(x) = \hat{\mathbf{e}}(A)$.

Restricting Lists by Sizes. Next, we show that these sizes can restrict possible members of lists of pendant elements:

LEMMA 5.1. *For a pendant element x , the possible pendant edge and all loops and half-edges of the list $\mathfrak{L}(x)$ have the same $\hat{\mathbf{v}}$ and $\hat{\mathbf{e}}$ as $\hat{\mathbf{v}}(x)$ and $\hat{\mathbf{e}}(x)$, respectively.*

Proof. The pendant element x corresponds to a block part of H . All members of $\mathfrak{L}(x)$ can be fully expanded to graphs isomorphic to this block part. Necessarily, these graphs contain the same number of vertices and edges as $\hat{\mathbf{v}}(x)$ and $\hat{\mathbf{e}}(x)$. \square

When $\mathfrak{L}(x)$ is computed, we can only consider quotients of the correct sizes, speeding up Algorithm 2. For pendant edges and loops, each belongs to lists of only one type of pendant elements by Lemma 4.5. This is not true for half-edges and for purpose of this section, the following is important:

COROLLARY 5.2. *Let x and y be two pendant elements.*

- (i) *If $\mathfrak{L}(x)$ and $\mathfrak{L}(y)$ share a half-edge, then $\hat{\mathbf{v}}(x) = \hat{\mathbf{v}}(y)$ and $\hat{\mathbf{e}}(x) = \hat{\mathbf{e}}(y)$.*
- (ii) *Let $\mathfrak{L}(x)$ contain a loop of a color c and a half-edge of a color c' . Then $\mathfrak{L}(y)$ cannot contain both the loop of the color c' and the half-edge of the color c .*

Proof. (i) Implied by Lemma 5.1.

(ii) Let $\mathfrak{L}(x)$ contain a loop e and $\mathfrak{L}(y)$ contain a half-edge h of the same color. Then $\hat{\mathbf{v}}(x) = \hat{\mathbf{v}}(e) + 1$ and $\hat{\mathbf{v}}(y) = \hat{\mathbf{v}}(e)/2 + 1$ for the vertices, and $\hat{\mathbf{e}}(x) = \hat{\mathbf{e}}(e)$ and $\hat{\mathbf{e}}(y) = \hat{\mathbf{e}}(e)/2$ for the edges. Therefore x corresponds

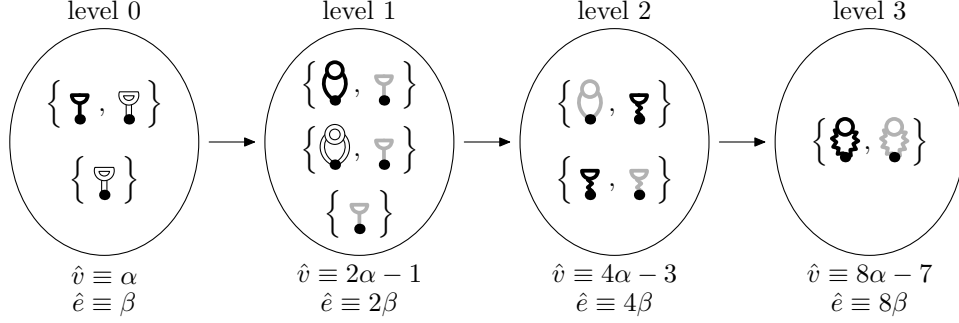


FIG. 5.4. A chain of pendant elements with four levels, which is the only chain in Fig. 5.1, for some α and β (we ignore multiplicities of pendant elements). Notice that quotients corresponding to one atom are placed in neighboring levels.

to a larger block part in H than y . By the same argument, we deduce that y corresponds to a larger block part in H than x , which gives a contradiction. \square

The property (i) relates half-edges together. The property (ii) states that there is a certain size hierarchy on the pendant elements discussed below.

Chains of Pendant Elements. Pendant elements can be partitioned into independent chains, each further partitioned into several levels. The level of size (α, β) consists of all pendant elements x having $\hat{v}(x) = \alpha$ and $\hat{e}(x) = \beta$. Each chain starts with the level 0 of some size (α, β) . Further, it contains the levels $m > 0$ of sizes $(2^m\alpha - (2^m - 1), 2^m\beta)$. See Fig. 5.4 for an example.

The key property is the following: if $\mathcal{L}(x)$ contains a half-edge of a color c and $\mathcal{L}(y)$ contains the loop of the same color c , then x belongs to a level m and y belongs to the level $m + 1$ of the same chain. A star block atom A can contain multiple chains, but different chains contain completely different colors in their lists, so they are completely independent.

Summary. We may partition S , D , and A according chains of pendant elements of A and test them separately. Therefore, we may assume that there is exactly one chain of pendant elements in A , and only the corresponding edges in D and half-edges in S .

5.3. Reduction to the IV-Matching Problem. The star block atom S contains a half-edge corresponding to the dipole D and let $\mathbf{H}'(S)$ be the set of the remaining half-edges corresponding to proper atoms. The dipole D has the following half-quotients Q . For each color class of an even size s , we choose an arbitrary integer ℓ such that $0 \leq \ell \leq \frac{s}{2}$, and attach ℓ loops and $s - 2\ell$ half-edges of this color in Q .

If these values s and ℓ are known for each color class, we can test existence of a perfect matching as described in the proof of Lemma 4.7. Since they are not known, we need to solve a generalization of perfect matching called IV-MATCHING which includes choosing of these values as a part of the problem.

Definition of the Problem. The input of IV-MATCHING gives the following bipartite graph B . We have $\mathbf{V}(B) = \mathbf{P}(A) \cup \mathbf{E}(D) \cup \mathbf{H}'(S)$. For $e \in \mathbf{E}(D)$ of a color c and $x \in \mathbf{P}(A)$, we have $ex \in \mathbf{E}(B)$ if and only if the half-edge or the loop of the color c belongs to the list $\mathcal{L}(x)$. We call the former case a *half-incidence* and the latter case a *loop-incidence*. Further, we have a *half-incidence* $hx \in \mathbf{E}(B)$ between $h \in \mathbf{H}'(S)$ of a color c and $x \in \mathbf{P}(A)$ if and only if the half-edge of the color c belongs to $\mathcal{L}(x)$.

We ask whether there exists a *spanning subgraph* B' of B , called an IV-subgraph of B , satisfying the following properties. Each component of connectivity of B' is a path of length one or two (corresponding to I and V in the name). Each $x \in \mathbf{P}(A)$ is in B' either half-incident to exactly one vertex in $\mathbf{E}(D) \cup \mathbf{H}'(S)$, or it is loop-incident to exactly two edges $e, e' \in \mathbf{E}(D)$ of the same color class. Further, each vertex of $\mathbf{E}(D) \cup \mathbf{H}'(S)$ is incident in B' to exactly one $x \in \mathbf{P}(A)$. See Fig. 5.5 for an example, with several additional properties which we discuss below.

Level Structure. The structure of levels of the chain of pendant elements transfers into the level structure of B . The part $\mathbf{P}(A)$ is partitioned into levels called *A levels*. Every half-edge $h \in \mathbf{H}'(S)$ is half-incident only to vertices of the *A level* m of the size $(\hat{v}(h), \hat{e}(h))$. Every edge $e \in \mathbf{E}(D)$ is half-incident only to vertices of the *A level* m of size $(\hat{v}(e)/2, \hat{e}(e)/2)$ and loop-incident only to vertices of the *A level* $m + 1$ of the size $(\hat{v}(e) - 1, \hat{e}(e))$. Therefore, we can define *S levels* for the part $\mathbf{E}(D) \cup \mathbf{H}'(S)$ such that an *S level* m contains

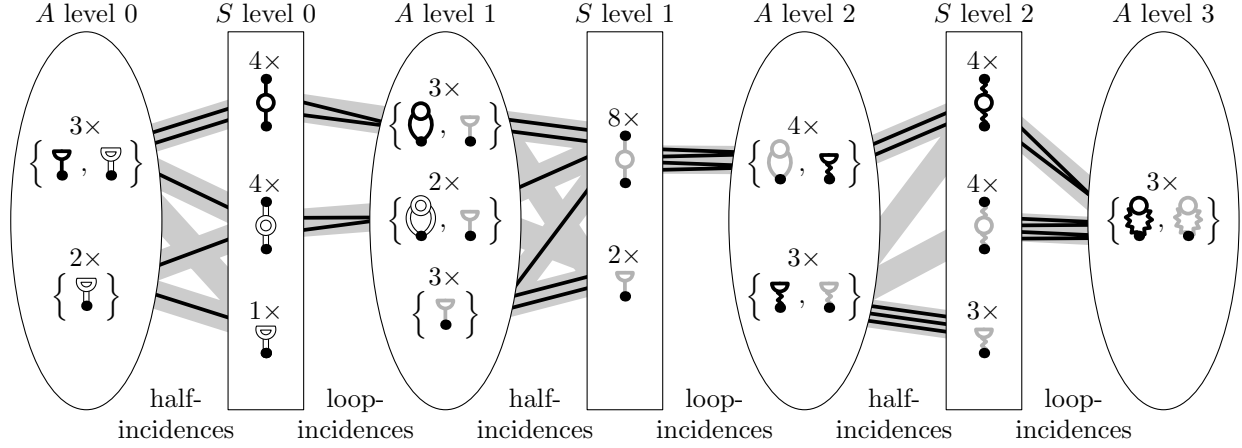


FIG. 5.5. The instance of the IV-MATCHING problem corresponding to the input A in Fig. 5.1 and the preprocessed star block atom S in Fig. 5.2. The edges $E(B)$ are depicted in gray and an IV-subgraph B' is highlighted in bold. We have I 's between half-incidences and V 's between loop-incidences. The part $P(A)$ is in ellipses, the other part $E(D) \cup H'(S)$ is in boxes. The lists $P(A)$, the edges $E(D)$ and the half-edges $H'(S)$ are depicted together with multiplicities.

all vertices half-incident to the A level m or loop-incident to the A level $m + 1$. If we depict all levels from left to right according to their order, alternating A levels and S levels, all edges of B go between consecutive levels as depicted in Fig. 5.5.

Clusters. We can view B as a cluster graph. In each S level, $E(D)$ and $H'(S)$ form clusters according to their color classes, called *edge clusters* and *half-edge clusters* respectively. In each A level, the pendant elements form *pendant element clusters* according to equivalence classes of their lists. (We note that two pendant elements with equal lists can correspond to non-isomorphic subgraphs in H . Then their lists contain only half-edges.) Two clusters are either completely adjacent, or not adjacent at all: the subgraph induced by the union of two clusters is either a complete bipartite graph, or contains no edges.

Each pendant element cluster may be loop-adjacent to several edge clusters. On the other hand, each edge cluster is loop-adjacent to at most one pendant element cluster: the loop of the corresponding color belongs to at most one isomorphism class of pendant elements by Lemma 4.5. There are no constraints for half-incidences between clusters.

Only Logarithmically Many Levels. The sizes of graphs are growing exponentially with the level number. Therefore, we have at most logarithmically many levels with respect to the size of the input graph H .

Complexity. Since IV-MATCHING can be used to solve the slow subroutine of Lemma 4.7, we get the following in relation to the meta-algorithm of Theorem 1.3:

PROPOSITION 5.3. *If the IV-MATCHING problem can be solved for logarithmically many levels in polynomial time, then we can modify the meta-algorithm of Theorem 1.3 to run in polynomial time as well.*

Unfortunately, Folwarczný and Knop [28] recently proved that this problem is NP-complete, even for two A levels. Nevertheless, we believe that the particular instances arising from the REGULARCOVER problem might be solvable in polynomial time and the properties described in this section might be useful for that.

Numbers of Edges Between Levels. We do not know how many half- and loop-incidences are in B' at each cluster, otherwise we could solve the problem directly by finding a perfect matching in a modified graph. On the other hand, these numbers are determined between consecutive A levels and S levels as follows. Let a_m be the number of pendant elements in the A level m and let s_m be the number of edges and half-edges in the S level m . Let B' be an IV-subgraph and let b_i and b'_i be the numbers of edges in B' between the A level i and the S level i , and between the S level i and the A level $i + 1$, respectively.

Every pendant element in the A level 0 is half-incident in B' to the S level 0, so $b_0 = a_0$. Therefore, the number of remaining vertices in the S level 0 is $s_0 - b_0$. These vertices are loop-incident in B' to the A level 1, so $b'_0 = s_0 - b_0$ and $a_1 - \frac{b'_0}{2}$ pendant elements remain in the A level 1. We can proceed in this way further, and we get the following inductive formulas:

$$b_i = a_i - \frac{b'_{i-1}}{2}, \quad \text{and} \quad b'_i = s_i - b_i.$$

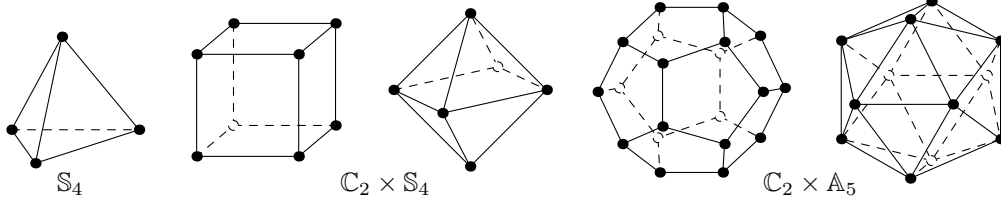


FIG. 6.1. The five platonic solids together with their automorphism groups.

Clearly, each number b'_i has to be even, otherwise no IV-subgraph exists.

6. Applying the Meta-algorithm to Planar Graphs. In this section, we discuss automorphism groups of 3-connected planar graphs and we show that the meta-algorithm of Theorem 1.3 applies to the class of planar graphs.

Automorphism Groups. A group is *spherical* if it is the group of the symmetries of a tiling of the sphere. The first class of spherical groups are the subgroups of the automorphism groups of the platonic solids, i.e., S_4 for the tetrahedron, $C_2 \times S_4$ for the cube and the octahedron, and $C_2 \times A_5$ for the dodecahedron and the icosahedron; see Fig. 6.1. The second class of spherical groups is formed by the infinite families C_n , D_n , $C_n \times C_2$, and $D_n \times C_2$.

A map \mathcal{M} is a 2-cell embedding of a graph G onto the sphere. A *rotation* at a vertex is a cyclic ordering of the edges incident with the vertex. An *angle* is a triple (v, e, e') where v is a vertex, and e and e' are two incident edges which are consecutive in the rotation at v or in the inverse rotation at v . An automorphism of a map is an automorphism of the graph which preserves the angles; in other words the rotations are preserved. For a 3-connected planar graph G , we have $\text{Aut}(G) \cong \text{Aut}(\mathcal{M})$ and it is a spherical group. For more details, see [26] and the references therein.

LEMMA 6.1. *For a 3-connected planar graph G , we can compute $\text{Aut}(G)$ in time $\mathcal{O}(v^2(G))$.*

Proof. Compute the unique map \mathcal{M} of G . There are $\mathcal{O}(v(G))$ angles in \mathcal{M} . We fix one angle (v, e, e') , and test for each other angle whether there is an automorphism mapping (v, e, e') to it. The key observation is that this partial mapping has a unique extension to a mapping compatible with the rotations of \mathcal{M} . We can just test in $\mathcal{O}(v(G))$ whether it is an automorphism. The total running time is $\mathcal{O}(v^2(G))$. \square

Properties (P1) to (P3). We are ready to establish the following:

LEMMA 6.2. *The class of planar graphs satisfies (P1) to (P3).*

Proof. The class of planar graphs clearly satisfies (P1). For (P2), Lemma 6.1 allows to compute $\text{Aut}(G)$ in time $\mathcal{O}(v^2(G))$. Since it is a spherical group, we can generate all linearly many subgroups and check which ones act semiregularly. The property (P3) holds for projectively planar graphs since LISTISO can be solved in time $\mathcal{O}(v^{5/2}(G))$ for graphs of bounded genus [41] (even for lists on both vertices and half-edges). \square

Proof. [Theorem 1.1] By Lemma 6.2, we can apply Theorem 1.3. \square

Half-quotients of Planar Proper Atoms. Let A be a planar proper atom. We know that $\text{Aut}(A)$ is a spherical group, and further each semiregular involution τ defining a half-quotient of A has to exchange the vertices of the boundary. We get two types of geometrically defined quotients, depicted in Fig. 6.2.

LEMMA 6.3 ([26], Lemma 5.6). *Let A be a planar proper atom and let $\partial A = \{u, v\}$. There are at most two half-quotients $A/\langle \tau \rangle$ where $\tau \in \text{Aut}(A)$ is an involutory semiregular automorphism transposing u and v :*

- (a) The rotational half-quotient – The involution τ is orientation preserving and $A/\langle \tau \rangle$ is planar with at most one half-edge.
- (b) The reflectional half-quotient – The involution τ is a reflection and $A/\langle \tau \rangle$ is planar with at least two half-edges.

7. Concluding Remarks. This paper is based on the structural results of [26], describing behaviour of regular graph covering with respect to 1-cuts and 2-cuts in G . In Theorem 1.3, we derive an FPT meta-algorithm for testing regular graph covers for \mathcal{C} -inputs G where \mathcal{C} is a class of graphs satisfying (P1) to (P3). In particular, this meta-algorithm is tailored for the class of planar graphs (Theorem 1.1).

When working with 3-connected decomposition, we described two subroutines we need to solve. First, we have rediscovered graph isomorphism restricted by lists, introduced by Lubiw [47], which lead to several

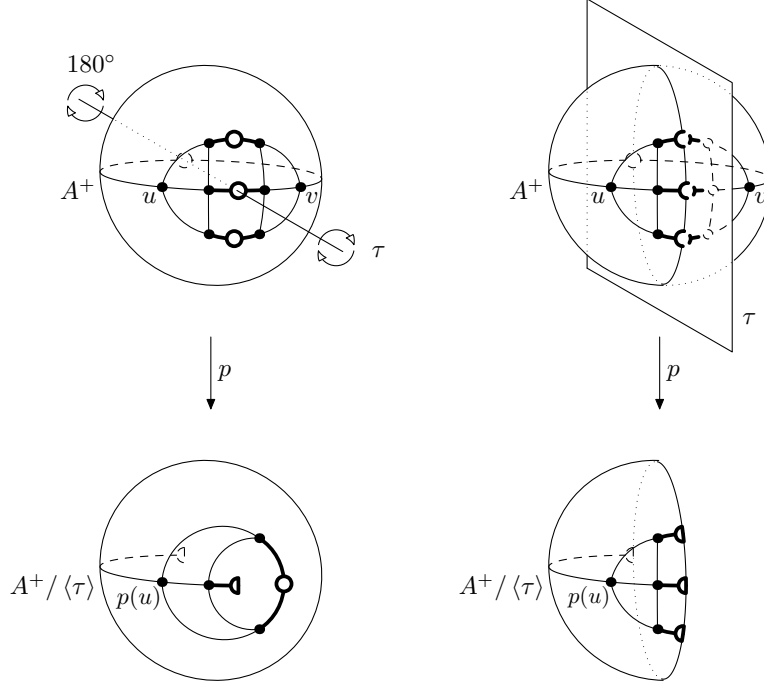


FIG. 6.2. The rotational quotient and reflectional quotient of a planar proper atom A with the added edge uv .

fruitful results in [41]. Second, we introduce a generalization of bipartite matching called the IV-MATCHING problem, proved to be NP-complete by Folwarcný and Knop [28].

We conclude by several remarks and open problems.

Running Time of The Meta-algorithm. We have omitted polynomial factors in the complexity since the main goal was to establish that REGULARCOVER can be solved in FTP time for \mathcal{C} -inputs G for \mathcal{C} satisfying (P1) to (P3). The degree of the polynomial depends on the complexity of polynomial time algorithms in (P2) and (P3).

We roughly estimate the degree of the polynomial for running time of the algorithm of Theorem 1.1 for planar graphs. Let $n = v(G) \geq v(H)$. For planar graphs, each primitive graph has $\mathcal{O}(n)$ quotients and each proper atom has at most two half-quotients. Therefore, the catalog contains $\mathcal{O}(n^2)$ atoms and quotients. Each catalog query can be answered in time $\mathcal{O}(n^4)$, and with a suitable canonization even in time $\mathcal{O}(n^3)$.

The reduction series can be computed in time $\mathcal{O}(n)$ by Hopcroft and Tarjan [37], the symmetry type of each atom can be determined in time $\mathcal{O}(n^2)$ by Lemma 6.1. When adding a proper atom to the catalog, we compute its at most two half-quotients in time $\mathcal{O}(n^2)$ and compute their reduction series in time $\mathcal{O}(n^4)$. Together with catalog queries, we can compute the reduction series and G_r in time $\mathcal{O}(n^5)$.

Next, we iterate over $\mathcal{O}(n)$ quotients H_r of G_r . For each, we compute the reduction series in time $\mathcal{O}(n^4)$. Next, we iterate over $\mathcal{O}(n)$ choices of the core in H . We compute the reduction series with lists where $\mathcal{O}(n)$ subroutines are called. The subroutine of Lemma 4.6 runs in time $\mathcal{O}(n^{9/2})$ since LISTISO takes time $\mathcal{O}(n^{5/2})$ [41] and we compare each non-star block atom A with $\mathcal{O}(n^2)$ candidates from the catalog. The subroutine of Lemma 4.7 runs in time $\mathcal{O}(n^2 2^{e(H)/2})$. The final test in Lemma 4.9 runs in time $\mathcal{O}(n^{5/2})$.

In total, the running time of the algorithm in Theorem 1.1 is $\mathcal{O}(n^5 \cdot (n^{5/2} + 2^{e(H)/2}))$. By a more careful analysis of the subroutines and possibly reordering them, it should be possible to decrease the degree little bit.

We did not try to optimize the factor $2^{e(H)/2}$. This estimate is certainly very rough and maybe some further techniques from parameterized complexity can be applied to solve IV-MATCHING faster. We believe that this approach should be followed only when we can prove that REGULARCOVER is NP-complete for planar inputs G . Also, to prove Theorem 1.3, it might be possible to design a simpler FPT algorithm running in time $\mathcal{O}^*(2^{e(H)/2})$. Our goal was to obtain as much understanding of the problem as possible, in order to construct a polynomial time algorithm. We failed with the subroutine of Lemma 4.7, but nevertheless

further structural results may be established and the problem may be solvable in polynomial time.

Possible Extensions of The Meta-algorithm. There are several possible natural extensions of the meta-algorithm. First, we can easily generalize it for input graph G and H with half-edges, directed edges and halvable edges, and also for colored graphs. Further, for a regular covering testing, one can prescribe a list $\mathcal{L}(u) \subseteq V(H)$ of allowed images of a regular covering projection p for each vertex $u \in V(G)$ such that $p(u) \in \mathcal{L}(u)$: the expandability testing subroutine can compute with these lists as well.

Complexity of Regular Graph Covering. By Lemmas 2.1 and 2.3, it follows that REGULARCOVER is GI-hard and belongs to NP. Its complexity remains an open problem:

PROBLEM 7.1. *What is the complexity of the REGULARCOVER problem?*

One possibility to attack this problem would be to prove that it is NP-hard, or to construct an efficient algorithm using an oracle for GRAPHISO. If REGULARCOVER is not NP-hard, another possibility is to prove that REGULARCOVER satisfies some properties which unlikely hold for any NP-hard problem. For instance for the graph isomorphism problem, there are currently three evidences that it is unlikely NP-complete: equivalence of existence and counting [6, 49], GRAPHISO belongs to coAM, so polynomial-hierarchy would collapse if GRAPHISO is NP-complete [29, 55], and GRAPHISO can be solved in subexponential time [8].

As a possible next direction of research, we suggest to attack classes of graphs close to planar graphs, for instance projective planar graphs or toroidal graphs. To do so, it seems that new techniques need to be built. Even the automorphism groups of projective planar graphs and toroidal graphs are not yet well understood.

It is natural to ask whether FPT running time of the meta-algorithm of Theorem 1.3 is needed:

PROBLEM 7.2. *Can the REGULARCOVER problem be solved in polynomial time for \mathcal{C} -inputs G where \mathcal{C} satisfies (P1) to (P3)? Can it be solved in polynomial time for planar inputs G ?*

The IV-Matching Problem. In Section 5, we have shown that the bottleneck of the algorithm of Theorem 1.3 reduces to a generalized matching problem called IV-MATCHING. Here, we describe a purely combinatorial formulation of the IV-MATCHING problem. This reformulation can be useful to understand the problem without regular covering and the structural results obtained in [26] and this paper.

The input of IV-MATCHING consists of a bipartite graph B with a partitioning V_1, \dots, V_ℓ of its vertices $V(B)$ which we call *levels*, with all edges between of consecutive levels V_i and V_{i+1} , for $i = 1, \dots, \ell - 1$. The levels V_1, V_3, \dots are called *odd* and the levels V_2, V_4, \dots *even*. Further each level V_i is partitioned into several *clusters*, each consisting of a few vertices with identical neighborhoods. There are three key properties:

- The incidences in B respect the clusters; between any two clusters the graph B induces either a complete bipartite graph, or an edge-less graph.
- Each cluster of an even level V_{2t} is incident with at most one cluster at V_{2t+1} .
- The incidences between the clusters of V_{2t-1} and V_{2t} can be arbitrary.

The problem IV-MATCHING asks whether there is a *spanning subgraph* B' called an IV-subgraph of B . Each component of connectivity of B' equals to a path of length one or two. Each vertex of an odd level V_{2t+1} is in B' adjacent either to exactly one vertex of V_{2t+2} , or to exactly two vertices of V_{2t} . Each vertex of an even level V_{2t} is adjacent to exactly one vertex of the levels $V_{2t-1} \cup V_{2t+1}$. In other words, from V_{2t-1} to V_{2t} the edges of B' form a matching, not necessarily perfect. From V_{2t} to V_{2t+1} , the edges of B' form independent V-shapes, with their centers in the level V_{2t+1} . Figure 7.1 shows an example.

In the conference version of this paper [25], we asked as an open problem what is the complexity of the IV-MATCHING problem. Recently, Folwarcný and Knop [28] answered this by proving that IV-MATCHING is strongly NP-hard even for $\ell = 3$. We note that this does not imply NP-hardness of REGULARCOVER, and it is possible that specific instances of IV-MATCHING arising from REGULARCOVER can be solved in polynomial time.

A Weaker Assumption (P2'). To make Theorem 1.3 a more natural generalization of Babai's algorithm [5] for graph isomorphism, it would be nice to replace (P2) with a weaker assumption:

- (P2') For a 3-connected graph $G \in \mathcal{C}$ with colored vertices and colored possibly directed edges and a graph H with colored vertices and colored possibly directed edges, half-edges and loops, we can test REGULARCOVER(G, H) in polynomial time.

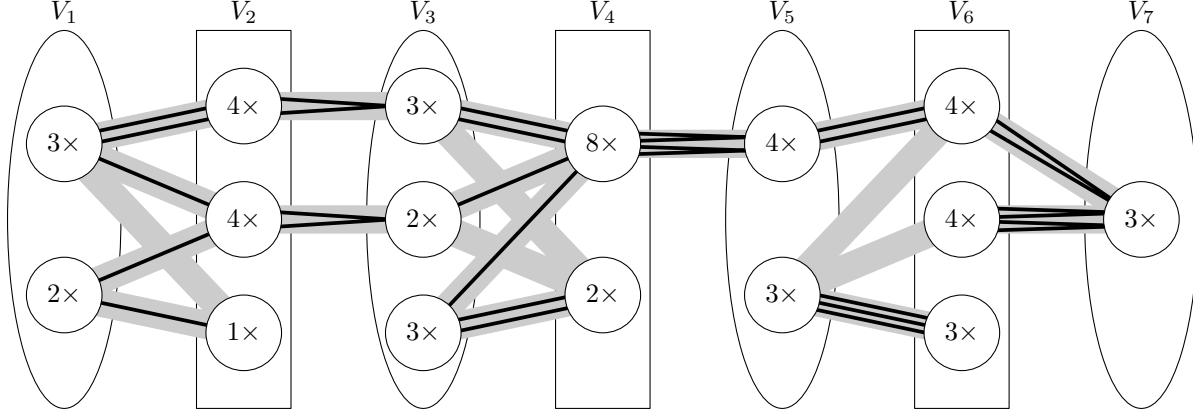


FIG. 7.1. An example input B , the clusters are depicted by circles together with their sizes. The odd levels are drawn in circles and the even ones in rectangles. The edges of B are depicted by gray lines between clusters representing complete bipartite graphs. One spanning subgraph B' solving the IV-MATCHING problem is depicted in bold.

It is an open problem whether our meta-algorithm can be modified for \mathcal{C} satisfying (P1), (P2') and (P3):³

PROBLEM 7.3. *Can the REGULARCOVER problem be solved in FPT time (with respect to the parameter $e(H)$) for \mathcal{C} -inputs G where \mathcal{C} satisfies (P1), (P2'), and (P3)?*

The modified algorithm would have to process both G and H simultaneously. For instance, it is not needed to decide whether a proper atom is halvable or symmetric. Also, we do not need to compute all half-quotients of a proper atom, we only need to test whether some subgraphs located in H are one of them. There are several issues with this approach which make this generalization a very tricky problem.

As illustrated in Fig. 4.5, a half-quotient of a proper atom might not be essentially 3-connected (but it is always essentially 2-connected if the proper atom is not a path). Therefore, we would locate Q' in H which is not a half-quotient of A . To test it using (P2), we would need to expand some proper atoms and dipoles in Q' to reach Q , but it is not clear which ones.

Even more involved is to avoid finding of all quotients $H_r = G_r/\Gamma_r$. Since H_r might consist of many blocks which might contain many proper atoms and dipoles, it is not clear how to locate it in H to test existence of a regular covering using (P2).

REFERENCES

- [1] J. ABELLO, M. R. FELLOWS, AND J. C. STILLWEIL, *On the complexity and combinatorics of covering finite complexes*, Australian Journal of Combinatorics, 4 (1991), pp. 103–112.
- [2] S. B. AKERS AND B. KRISHNAMURTHY, *On group graphs and their fault tolerance*, IEEE Trans. Comput., 36 (1987), pp. 885–888.
- [3] D. ANGLUIN, *Local and global properties in networks of processors*, in ACM Symposium on Theory of Computing, ACM, 1980, pp. 82–93.
- [4] D. ANGLUIN AND A. GARDINER, *Finite common coverings of pairs of regular graphs*, J. Combin. Theory Ser. B, 30 (1981), pp. 184–187.
- [5] L. BABAI, *Automorphism groups of planar graphs II*, in Infinite and finite sets (Proc. Conf. Keszthely, Hungary, 1973) Bolyai-North-Holland, 1975, pp. 29–84.
- [6] ———, *On the isomorphism problem*, (1977).
- [7] ———, *Automorphism groups, isomorphism, reconstruction*, in Handbook of combinatorics (vol. 2), MIT Press, 1996, pp. 1447–1540.
- [8] ———, *Graph isomorphism in quasipolynomial time*, in STOC, 2016.
- [9] R. BAR-YEHUDA AND T. ETZION, *Connections between two cycles – a new design of dense processor interconnection networks*, Discrete Applied Mathematics, 37–38 (1992), pp. 29–43.
- [10] N. BIGGS, *Algebraic graph theory*, Cambridge University Press, 1993.
- [11] O. BÍLKA, J. JIRÁSEK, P. KLAVÍK, M. TANCER, AND J. VOLEC, *On the complexity of planar covering of small graphs*, in WG 2011, vol. 6986 of Lecture Notes in Computer Science, 2011, pp. 83–94.
- [12] H. L. BODLAENDER, *The classification of coverings of processor networks*, Journal of Parallel and Distributed Computing, 6 (1989), pp. 166–182.
- [13] L. CAMPBELL, *Dense group networks*, Discrete Applied Mathematics, 37–38 (1992), pp. 65–71.

³Even more natural would be to replace (P3) with (P3*), but this problem seems even more tricky.

- [14] L. CAMPBELL, G. E. CARLSSON, M. J. DINNEEN, V. FABER, M. R. FELLOWS, M. A. LANGSTON, J. W. MOORE, A. P. MULLHAUPT, AND H. B. SEXTON, *Small diameter symmetric networks from linear groups*, IEEE Trans. Comput., 41 (1992), pp. 218–220.
- [15] G. E. CARLSSON, J. E. CRUTHIRDS, H. B. SEXTON, AND C. G. WRIGHT, *Interconnection networks based on a generalization of cube-connected cycles*, IEEE Trans. Comput., 100 (1985), pp. 769–772.
- [16] A. CAYLEY, *The theory of groups: Graphical representation*, Amer. J. Math., 1 (1878), pp. 174–176.
- [17] S. CHAPLICK, J. FIALA, P. VAN’T HOF, D. PAULUSMA, AND M. TESAR, *Locally constrained homomorphisms on graphs of bounded treewidth and bounded degree*, Theoretical Computer Science, 590 (2015), pp. 86–95.
- [18] C. J. COLBOURN AND K. S. BOOTH, *Linear times automorphism algorithms for trees, interval graphs, and planar graphs*, SIAM J. Comput., 10 (1981), pp. 203–225.
- [19] W. H. CUNNINGHAM AND J. EDMONDS, *A combinatorial decomposition theory*, Canad. J. Math., 32 (1980), pp. 734–765.
- [20] P. ERDŐS, S. FAJTLOWICZ, AND A. J. HOFFMAN, *Maximum degree in graphs of diameter 2*, Networks, 10 (1980), pp. 87–90.
- [21] S. EVDOKIMOV AND I. PONOMARENKO, *Circulant graphs: recognizing and isomorphism testing in polynomial time*, St. Petersburg Mathematical Journal, 15 (2004), pp. 813–835.
- [22] H. M. FARKAS AND I. KRA, *Riemann surfaces*, in Riemann Surfaces, vol. 71 of Graduate Texts in Mathematics, 1992, pp. 9–31.
- [23] T. FEDER AND M. Y. VARDI, *The computational structure of monotone monadic snp and constraint satisfaction: A study through datalog and group theory*, SIAM Journal on Computing, 28 (1998), pp. 57–104.
- [24] J. FIALA, *Note on the computational complexity of covering regular graphs*, in 9th Annual Conference of Doctoral Students, WDS’00, Matfyzpress, 2000, pp. 89–90.
- [25] J. FIALA, P. KLAVÍK, J. KRATOCHVÍL, AND R. NEDELA, *Algorithmic aspects of regular graph covers with applications to planar graphs*, in Automata, Languages, and Programming, 41st International Colloquium, ICALP 2014, vol. 8572 of Lecture Notes in Computer Science, 2014, pp. 489–501.
- [26] ———, *3-connected reduction for regular graph covers*, CoRR, abs/1503.06556 (2016).
- [27] J. FIALA AND J. KRATOCHVÍL, *Locally constrained graph homomorphisms structure, complexity, and applications*, Computer Science Review, 2 (2008), pp. 97–111.
- [28] L. FOLWARCZNY AND D. KNOP, *IV-matching is strongly NP-hard*, CoRR, abs/1506.08388 (2015).
- [29] O. GOLDBREICH, S. MICALI, AND A. WIGDERSON, *Proofs that yield nothing but their validity and a methodology of cryptographic protocol design*, in FOCS, vol. 86, 1986, pp. 174–187.
- [30] D. M. GOLDSCHMIDT, *Automorphisms of trivalent graphs*, Annals of Mathematics, 111 (1980), pp. 377–406.
- [31] J. L. GROSS AND T. W. TUCKER, *Topological graph theory*, Courier Dover Publications, 2001.
- [32] D. GUO, J. WU, H. CHEN, AND X. LUO, *Moore: An extendable peer-to-peer network based on incomplete Kautz digraph with constant degree*, in INFOCOM 2007. 26th IEEE International Conference on Computer Communications. IEEE, 2007, pp. 821–829.
- [33] Z. HEDRLÍN AND A. PULTR, *On full embeddings of categories of algebras*, Illinois Journal of Mathematics, 10 (1966), pp. 392–406.
- [34] P. HELL AND J. NEŠETŘIL, *On the complexity of H-coloring*, J. Combin. Theory Ser. B, 48 (1990), pp. 92–110.
- [35] A. J. HOFFMAN AND R. R. SINGLETON, *On moore graphs with diameters 2 and 3*, IBM Journal of Research and Development, 4 (1960), pp. 497–504.
- [36] I. HOLYER, *The np-completeness of edge-coloring*, SIAM Journal on Computing, 10 (1981), pp. 718–720.
- [37] J. E. HOPCROFT AND R. E. TARJAN, *Dividing a graph into triconnected components*, SIAM Journal on Computing, 2 (1973), pp. 135–158.
- [38] J. E. HOPCROFT AND J. WONG, *Linear time algorithm for isomorphism of planar graphs (preliminary report)*, in Proceedings of the sixth annual ACM symposium on Theory of computing, ACM, 1974, pp. 172–184.
- [39] C. JORDAN, *Sur les assemblages de lignes.*, Journal für die reine und angewandte Mathematik, 70 (1869), pp. 185–190.
- [40] M. O. KATANAIEV, *All universal coverings of two-dimensional gravity with torsion*, Journal of mathematical physics, 34 (1993), pp. 700–736.
- [41] P. KLAVÍK, D. KNOP, AND P. ZEMAN, *Graph isomorphism restricted by lists*, CoRR, abs/1607.03918 (2016).
- [42] P. KLAVÍK, R. NEDELA, AND P. ZEMAN, *Constructive approach to automorphism groups of planar graphs*, CoRR, abs/1506.06488 (2016).
- [43] P. KLAVÍK AND P. ZEMAN, *Automorphism groups of geometrically represented graphs*, in STACS 2015, vol. 30 of LIPIcs, 2015, pp. 540–553.
- [44] ———, *Automorphism groups of geometrically represented graphs*, CoRR, abs/1407.2136 (2015).
- [45] J. KRATOCHVÍL, A. PROSKUROWSKI, AND J. A. TELLE, *Covering regular graphs*, J. Comb. Theory Ser. B, 71 (1997), pp. 1–16.
- [46] D. LICHTENSTEIN, *Isomorphism for graphs embeddable on the projective plane*, in ACM Symposium on Theory of Computing, STOC ’80, 1980, pp. 218–224.
- [47] A. LUBIW, *Some NP-complete problems similar to graph isomorphism*, SIAM Journal on Computing, 10 (1981), pp. 11–21.
- [48] E. M. LUKS, *Isomorphism of graphs of bounded valence can be tested in polynomial time*, Journal of Computer and System Sciences, 25 (1982), pp. 42–65.
- [49] R. MATHON, *A note on the graph isomorphism counting problem*, Information Processing Letters, 8 (1979), pp. 131–132.
- [50] B. D. MCKAY, M. MILLER, AND J. ŠIRÁŇ, *A note on large graphs of diameter two and given maximum degree*, J. Combin. Theory Ser. B, 74 (1998), pp. 110–118.
- [51] M. MILLER AND J. ŠIRÁŇ, *Moore graphs and beyond: A survey of the degree/diameter problem*, Electronic Journal of Combinatorics, 61 (2005), pp. 1–63.
- [52] S. NEGAMI, *The spherical genus and virtually planar graphs*, Discrete Mathematics, 70 (1988), pp. 159–168.
- [53] G. RINGEL AND J. YOUNGS, *Solution of the Heawood map-coloring problem*, Proceedings of the National Academy of Sciences of the United States of America, 60 (1968), pp. 438–445.
- [54] J. J. ROTMAN, *An Introduction to the Theory of Groups*, Graduate Texts in Mathematics, Springer, 1994.

- [55] U. SCHÖNING, *Graph isomorphism is in the low hierarchy*, Journal of Computer and System Sciences, 37 (1988), pp. 312–323.
- [56] B.A. TRAKHTENBROT, *Towards a theory of non-repeating contact schemes*, Trudi Mat. Inst. Akad. Nauk SSSR, 51 (1958), pp. 226–269.
- [57] W. T. TUTTE, *Connectivity in graphs*, vol. 15, University of Toronto Press, 1966.
- [58] J. ŠIAGIOVÁ, *A note on the McKay-Miller-Širáň graphs*, J. Combin. Theory Ser. B, 81 (2001), pp. 205–208.
- [59] T.R.S. WALSH, *Counting unlabeled three-connected and homeomorphically irreducible two-connected graphs*, J. Combin. Theory B, 32 (1982), pp. 12–32.
- [60] S. ZHOU, *A class of arc-transitive Cayley graphs as models for interconnection networks*, SIAM Journal on Discrete Mathematics, 23 (2009), pp. 694–714.